ANTICYCLOTOMIC IWASAWA INVARIANTS AND CONGRUENCES OF MODULAR FORMS

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Abstract. The main purpose of this article is to examine how congruences between Hecke eigensystems of modular forms affect the Iwasawa invariants of their anticyclotomic p-adic L-functions. We apply Greenberg-Vatsal and Emerton-Pollack-Weston's ideas on the variation of Iwasawa invariants under congruences to the anticyclotomic setting. As an application, we establish infinitely many new examples of the anticyclotomic main conjecture for modular forms, which cannot be treated by Skinner-Urban's work. An explicit example is given.

1. Introduction

1.1. **Overview.** Iwasawa theory of elliptic curves over the anticyclotomic $\mathbb{Z}_p$-extenstion of an imaginary quadratic field has its own arithmetic interest due to the extensive use of Heegner points, which have played an important role in attacking Birch and Swinnerton-Dyer conjecture. The essence of Iwasawa theory is encoded in the Iwasawa main conjectures. The Iwasawa main conjecture for this setting is proven under rather *strict* conditions due to Bertolini-Darmon [BD05] and Skinner-Urban [SU14].

We establish an anticyclotomic analogue of Greenberg-Vatsal's result [GV00] on the behavior of Iwasawa invariants of p-adic L-functions of modular forms under congruences. The main idea is to examine how congruences between Hecke eigensystems of automorphic forms on a definite quaternion algebra affect the Iwasawa invariants of their anticyclotomic p-adic L-functions by calculating a formula for λ-invariants of Euler factors at a prime ℓ which splits in the imaginary quadratic field and is prime to p.

Applying the main result of this article, we obtain infinitely many new examples of the anticyclotomic main conjecture of Iwasawa theory for modular forms, with more *relaxed* conditions following the philosophy of congruences developed by Vatsal [Vat99], Greenberg-Vatsal [GV00], Emerton-Pollack-Weston [EPW06], and Greenberg [Gre10]. We adapt and refine the setup of Pollack-Weston's generalization [PW11] of Bertolini-Darmon's Euler system divisibility of the anticyclotomic main conjecture of elliptic curves [BD05].

Note that our setup is more general than the setup of Skinner-Urban's work in the following sense.

- The prime p does not necessarily split in the fixed imaginary quadratic field.
- The ramification condition on the residual Galois representation is weaker than that of Skinner-Urban.

A subsequent paper [CKL] deals with the higher weight case via Hida theory for quaternion algebras and big Heegner point construction of two variable p-adic L-functions. See [CH16], [CH15] for the higher weight case with certain restrictions.

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1.2. **Statement of the main result.** Let p be a rational prime ≥ 5 and fix embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let f be a newform of weight 2 of level $\Gamma_0(N)$ which is good ordinary at p. Write N_f for the level when we need to specify. Let F be a finite extension of the Hecke field of f over $\mathbb{Q}_p$ and $\mathcal{O}$ be its ring of integers, and $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$ lying over p (which is compatible with the choice of the embedding ι_p). Let ϖ be a uniformizer, and $\mathbb{F}$ be the residue field of F.

Let $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(F)$ be the $\mathfrak{p}$-adic Galois representation attached to f where V_f is the 2-dimensional F-vector space. Assume that its mod $\mathfrak{p}$ reduction $\overline{\rho}_f$ is irreducible. Under this irreducibility condition, the mod ϖ reduction $\overline{\rho}_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{\mathbb{F}}(T_f/\varpi T_f) \simeq \mathrm{GL}_2(\mathbb{F})$ is uniquely determined up to isomorphism where T_f is a Galois-stable lattice over $\mathcal{O}$ in V_f. Write $A_f := V_f/T_f$.

Let K be an imaginary quadratic field with $(\mathrm{disc}(K), pN) = 1$ and K_∞ be the anticyclotomic $\mathbb{Z}_p$-extension of K. Then we have decomposition $N = N^+ \cdot N^-$ where a prime divisor of N^+ splits in K and a prime divisor of N^- is inert in K.

Assumption 1.1. *Assume that N^- is square-free and the product of an* odd *number of primes.*

By this parity condition, the sign of functional equation of the Rankin-Selberg L-function of f and K is $+1$.

Under this setting, we are able to consider the anticyclotomic p-adic L-function $L_p(K_\infty, f)$ à la Bertolini-Darmon ([BD96], [BD05]), which interpolates the algebraic part of L-values at $s = 1$ of the Rankin-Selberg convolution of the newform f and the theta series associated to a ring class character of p-power conductor over the imaginary quadratic field K. Note that its nontriviality is proved in [Vat02, Theorem 2.11].

Let K_∞ be the anticyclotomic $\mathbb{Z}_p$-extension of K and $\Gamma_\infty = \mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p$. Write K_n for the unique subfield of K_∞ such that $\mathrm{Gal}(K_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$. Let $\Lambda = \mathcal{O}[\![\Gamma_\infty]\!]$ be the anticyclotomic Iwasawa algebra over $\mathcal{O}$.

The anticyclotomic Iwasawa main conjecture predicts a deep relation between anticyclotomic p-adic L-functions and anticyclotomic Selmer groups.

Conjecture 1.2. (Anticyclotomic Iwasawa main conjecture) *The following statements hold:*

(1) $\mathrm{Sel}(K_\infty, A_f)$ *is Λ-cotorsion, and*
(2) *the equality of ideals of Λ*
$$(L_p(K_\infty, f)) = \mathrm{char}_\Lambda(\mathrm{Sel}(K_\infty, A_f)^\vee)$$
holds,

where the Selmer group here is the minimal *Selmer groups as in [PW11] and $(-)^\vee$ is the Pontryagin dual.*

From now on, we assume that f is not congruent to any p-newform; in other words,

Assumption 1.3. *Assume that $a_p(f) \not\equiv \pm 1 \pmod{p}$.*

Remark 1.4. This "non-anomalous" condition on $a_p(f)$ indicates that f is *not* congruent to any newform of level pN. If $a_p(f) \equiv \pm 1 \pmod{p}$, then the mod p reduction of the p-adic L-function ("mod p L-value") vanishes via the interpolation formula ([CH16, Theorem A]) due to mod p vanishing of the Euler factor at p. In other words, this mod p anomalous condition increases μ-invariants. It can be also regarded as a "mod p exceptional zero phenomena." This condition is also required to prove the second explicit reciprocity law à la [BD05] because Ihara's lemma for Shimura curves ([DT94, Theorem 2]) is crucially used in the proof. See [CH15, Lemma 5.3] for detail. The condition $p \geq 5$ is required to use the level raising technique of Diamond-Talyor. See [DT94, Theorem A (page 436)].

Let $\overline{\rho} \simeq \overline{\rho}_f$ be a residual modular Galois representation and $N(\overline{\rho})$ be its tame conductor. Then, as we did before, we decompose
$$N(\overline{\rho}) = N(\overline{\rho})^+ \cdot N(\overline{\rho})^-$$
using the same K.

We define a rather stringent condition first.

Definition 1.5. ($N^\pm$-minimality) Let f be a newform of weight 2 and level $\Gamma_0(N)$ with residual representation $\overline{\rho}$. The form f is $N^\pm$**-minimal** if $N^\pm = N(\overline{\rho})^\pm$, respectively.

Condition CR is a relaxation of N^--minimality.

Definition 1.6. (Condition CR: [Hel07, Definition 6.4], [PW11, Page 1354]) Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ be a modular Galois representation where $\mathbb{F}$ is a finite field of characteristic $p \geq 5$. Let N^- be a square-free product of an odd number of primes which is divisible by $N(\overline{\rho})^-$. The pair $(\overline{\rho}, N^-)$ satisfies **condition CR** if

- $\overline{\rho}$ is irreducible, and
- if $q \mid N^-$ and $q \equiv \pm 1 \pmod{p}$, then $\overline{\rho}$ is ramified at q.

We need a condition on the residual image for application to the main conjecture.

Definition 1.7. (Big image) The residual representation $\overline{\rho}$ has **big image** if the image of $\overline{\rho}$ contains a conjugate of $\mathrm{GL}_2(\mathbb{F}_p)$.

Remark 1.8. Condition CR refines "hypothesis CR" in [PW11], which originates from the idea to *control* the multiplicities of Galois representations arising from Jacobians of Shimura curves. See [Hel07] for detail. In [PW11], hypothesis CR requires that the residual representation is *surjective*, but it is too restrictive since the determinant of the representation is only the cyclotomic character. Moreover, it turns out that the surjectivity is *not* required due to a group-theoretic argument of Chida-Hsieh unless $\mathbb{F} = \mathbb{F}_5$. See [CH15, Lemma 6.1, Lemma 6.2 and Theorem 6.3] for detail.

Then the Euler system method provides the one sided divisibility of the main conjecture.

Theorem 1.9. (Euler system divisibility [BD05], [PW11, Theorem 4.1], [KPW15]) *Suppose that $(\overline{\rho}_f, N^-_f)$ satisfies condition CR and $\overline{\rho}_f$ has big image. Then we have:*

1. $\mathrm{Sel}(K_\infty, A_f)$ *is Λ-cotorsion, and*
2. *The divisibility of ideals*
$$\mathrm{char}_\Lambda(\mathrm{Sel}(K_\infty, A_f)^\vee) \mid (L_p(K_\infty, f))$$
holds.

Remark 1.10. (on the Euler system divisibility and congruences) The first proof of the Euler system divisibility is given in [BD05]. However, their work is not relevant to the context of congruences because their assumption does not allow any congruence ("p-isolated condition"). Moreover, Pollack-Weston's generalization [PW11] still requires the N^+-minimality for the freeness of compactified Selmer groups. These restrictions are removed introducing condition CR with an Iwasawa-theoretic argument in [KPW15].

The Eisenstein congruence method à la Ribet implies the opposite divisibility.

Theorem 1.11. (Eisenstein congruence divisibility [SU14]) *If $\overline{\rho}_f$ is irreducible and f is N^--minimal, we have the following divisibility.*
$$(L_p(K_\infty, f)) \mid \mathrm{char}_\Lambda(\mathrm{Sel}(K_\infty, A_f)^\vee)$$

Remark 1.12. The N^--minimality condition is more strict than condition CR. Thus, we have the equality of the anticyclotomic Iwasawa main conjecture assuming the N^--minimality. We do *not* appeal to the Eisenstein congruence divisibility in this article.

Our major purpose in this article is to investigate the behavior of anticyclotomic Iwasawa invariants. First, we recall Vatsal's result on anticyclotomic μ-invariants.

Theorem 1.13. ([Vat03, Theorem 1.1 and Proposition 4.7], [PW11, Theorem 2.3]) *Assume that $\overline{\rho}_f$ is irreducible and f satisfies Assumption 1.3. Then the μ-invariant of anticyclotomic p-adic L-function of f vanishes.*

Remark 1.14. (on vanishing of analytic μ-invariants)

- See [PW11, Theorem 2.3 and Remark 2.4] for the relation between analytic μ-invariants and congruence numbers.
- The Euler system divisibility directly implies that the corresponding algebraic μ-invariants also vanish.
- Under vanishing of μ-invariants, the λ-invariant of $L_p(K_\infty, f)$ can be detected by looking at "the mod p L-value" $L_p(K_\infty, f) \pmod{\varpi}$. See [EPW06, §1.4].
- Vanishing of μ-invariants is still an open problem in the cyclotomic case and is assumed in [GV00] and [EPW06].

Our main theorem describes the behavior of λ-invaraints of anticyclotomic p-adic L-functions under congruences. The rest part of this section is very rough, but the precise statements will be given in §6.

Main Theorem. **(Theorem 6.1)** *If two newforms of weight 2 are congruent and satisfy condition CR with the same N^-, then the difference of λ-invariants of their anticyclotomic p-adic L-functions can be calculated explicitly in terms of "Iwasawa-theoretic Euler factors".*

The Iwasawa-theoretic Euler factors, defined in §4, interpolate Euler factors twisted by anticyclotomic characters. There is an algebraic counterpart of the main theorem.

Theorem 1.15. ([PW11, Theorem 7.1], **Theorem 6.5)** *If two newforms of weight 2 are congruent and satisfy condition CR with the same N^-, then the difference of λ-invariants of their anticyclotomic Selmer groups can be calculated explicitly in terms of local cohomologies.*

Remark 1.16. (analytic vs. algebraic) The argument of the algebraic counterpart is completely formal. Comparing local conditions of primitive and non-primitive Selmer groups directly deduces the conclusion. On the other hand, the analytic side is more subtle since p-adic L-functions are not described in terms of local data. The key reason would be that p-adic L-functions do *not* admit Euler products. For instance, algebraic μ-invariants can be described in terms of local data, Tamagawa exponents, but analytic μ-invariants are only able to be described in terms of congruence numbers, which are global data. See [PW11, §6.5, 6.6]. Moreover, removing Euler factors at inert primes changes the sign of Rankin L-functions, which is also a global information. See Remark 5.2.(2). These phenomena indicate that the analytic side is more mysterious in the context of congruences of modular forms.

In §4, we precisely match up Iwasawa-theoretic Euler factors and local cohomologies. As a result, we can prove a relative statement of the anticyclotomic Iwasawa main conjecture for modular forms combining with the Euler system divisibility.

Corollary 1.17. **(Corollary 6.7)** *If two newforms of weight 2 are congruent and satisfy condition CR with the same N^-, then the anticyclotomic Iwasawa main conjecture for one form implies the anticyclotomic Iwasawa main conjecture for the other form.*

1.3. **Organization.** In §2, we review the basic notion of automorphic forms on a definite quaternion algebra, Jacquet-Langlnads correspondence, and construction of p-adic L-functions. In §3, we prove a mod p multiplicity one result for automorphic forms on a definite quaternion algebra consulting condition CR. In §4, we introduce Iwasawa-theoretic Euler factors and make a precise connection between Iwasawa-theoretic Euler factors and local cohomologies. In §5, we investigate the stability of normalization of automorphic forms stripping out Iwasawa-theoretic Euler factors. In §6, we prove the main theorem putting it all together and discuss the application of the main theorem to the main conjecture. At the end (§7), we give an explicit example. In Appendix A, we study Gross periods in more detail.

2. Automorphic forms on a definite quaternion algebra and p-adic L-functions

2.1. **Integral Jacquet-Langlands correspondence.** Let B be the definite quaternion algebra over $\mathbb{Q}$ of discriminant N^- and R be an oriented Eichler order of level N^+ in B. For any abelian group A, $\widehat{A} := A \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Then we obtain an **automorphic form on a definite quaternion algebra**

$$\phi_f : B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \to \mathbb{C}$$

via classical Jacquet-Langlands correspondence. We are able to normalize ϕ_f to take values in $\overline{\mathbb{Q}}$ since the double coset is finite ([Vig80, §5.B of chapitre III]) and it is an eigenform. Using the embedding ι_p, we have p-adically integral normalization $\phi_f : B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \to \mathcal{O}$ with $\phi_f(B^\times \backslash \widehat{B}^\times / \widehat{R}^\times) \not\equiv 0 \pmod{\varpi}$. For a detailed description of this normalization, see [PW11, §2].

We denote the space of $\mathcal{O}$-valued normalized automorphic forms by $M_2^{N^-}(K_0(N^+), \mathcal{O})$ and the subspace which the rank 1 subspace generated by the constant function is removed by $S_2^{N^-}(K_0(N^+), \mathcal{O})$. Note that the constant function in $M_2(B^\times \backslash \widehat{B}^\times / \widehat{R}^\times, \mathcal{O})$ is regarded as "Eisenstein series" since the corresponding residual representation is reducible. Let $\mathbb{T}^{N^-}(K_0(N^+))_{\mathcal{O}}$ be the full Hecke algebra over $\mathcal{O}$ faithfully acting on $S_2^{N^-}(K_0(N^+), \mathcal{O})$.

Theorem 2.1. (Jacquet-Langlands) *There is a (non-canonical) $\mathbb{T}^{N^-}(K_0(N^+))_{\mathcal{O}}$-equivariant isomorphism*

$$S_2(\Gamma_0(N),\mathcal{O})^{N^-\text{-}new} \simeq S_2^{N^-}(K_0(N^+),\mathcal{O}).$$

Proof. The statement with $\mathbb{C}$-coefficients can be found in [Hid04, §4.3.2] and [Hid06, §2.3.6]. With the fixed normalizations on both sides, we obtain the conclusion. □

Remark 2.2. (on the meaning of this integral correspondence) This integral Jacquet-Langlands correspondence is *ad hoc*. In some sense, we define our normalization to obtain this integral correspondence. This normalization is *directly* related to the analytic μ-invariant and *conjecturally* related to the congruence number. For a more meaningful integral Jacquet-Langlands correspondence, the connection between this normalization and the congruence number should be clarified. Under condition CR, the precise connection is proved in [PW11, Proposition 6.7].

2.2. **Bruhat-Tits trees, degeneracy maps and p-stabilization.** Let ℓ be an *arbitrary* prime not dividing N. (*e.g.* ℓ can be p.) Let $B_\ell := B\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$ and $R_\ell := R\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$. We consider double cosets $B^\times\backslash\widehat{B}^\times/\widehat{R}^\times$ and $B^\times\backslash\widehat{B}^\times/\widehat{R_0(\ell)}^\times$ where $R_0(\ell^n)$ is an oriented Eichler order of level $\ell^n N^+$ contained in R. In other words, $R_0(\ell^n)_\ell$ in $R_\ell\simeq \mathrm{M}_2(\mathbb{Z}_\ell)$ is isomorphic to $\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod{\ell^n}\}$ in $\mathrm{M}_2(\mathbb{Z}_\ell)$.

These double coset spaces admit a geometric and combinatorial description in terms of the Bruhat-Tits tree BT_ℓ for $\mathrm{PGL}_2(\mathbb{Q}_\ell)(= B_\ell^\times/\mathbb{Q}_\ell^\times)$ via strong approximation theorem ([Vig80, §4, Chapitre III] and [Dar04, Theorem 4.5]). We have the following canonical identification:

$$\begin{array}{lcl} \Gamma\backslash\mathcal{V}(BT_\ell) = R[\tfrac{1}{\ell}]^\times\backslash B_\ell^\times/R_\ell^\times & \xrightarrow{\simeq} & B^\times\backslash\widehat{B}^\times/\widehat{R}^\times \\ \qquad\qquad [b_\ell] & \longmapsto & [\cdots,1,b_\ell,1,\cdots] \\ \Gamma\backslash\vec{\mathcal{E}}(BT_\ell) = R[\tfrac{1}{\ell}]^\times\backslash B_\ell^\times/R_0(\ell)_\ell^\times & \xrightarrow{\simeq} & B^\times\backslash\widehat{B}^\times/\widehat{R_0(\ell)}^\times \\ \qquad\qquad [b_\ell] & \longmapsto & [\cdots,1,b_\ell,1,\cdots] \end{array} \tag{1}$$

where $\Gamma := R[\frac{1}{\ell}]^\times/\mathbb{Z}[\frac{1}{\ell}]^\times$ is a discrete subgroup of $B_\ell^\times/\mathbb{Q}_\ell^\times$, $\mathcal{V}(BT_\ell)$ is the set of the vertices of the Bruhat-Tits tree, $\vec{\mathcal{E}}(BT_\ell)$ is the set of the oriented edges of the Bruhat-Tits tree, and the adelic element $[\cdots,1,b_\ell,1,\cdots]$ is only nontrivial at ℓ.

We recall the degeneracy maps.

Definition 2.3. (Degeneracy maps at ℓ) We have the following degeneracy maps

$$\begin{aligned} \pi_{\ell,1} : B^\times\backslash\widehat{B}^\times/\widehat{R_0(\ell)}^\times &\to B^\times\backslash\widehat{B}^\times/\widehat{R}^\times \\ [\cdots,b_\ell,\cdots] &\mapsto [\cdots,b_\ell,\cdots] \end{aligned}$$

$$\begin{aligned} \pi_{\ell,2} : B^\times\backslash\widehat{B}^\times/\widehat{R_0(\ell)}^\times &\to B^\times\backslash\widehat{B}^\times/\widehat{R}^\times \\ [\cdots,b_\ell,\cdots] &\mapsto [\cdots,b_\ell\left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right),\cdots] \end{aligned}$$

where $b_\ell\in\mathrm{GL}_2(\mathbb{Q}_\ell)\simeq B_\ell^\times$ is an element in the ℓ-th slot.

We use the degeneracy maps at p to define the p-stabilization of automorphic forms.

Definition 2.4. The **p-stabilized automorphic form $\phi_f^{(\alpha_p)}$ of ϕ_f** is defined by

$$\phi_f^{(\alpha_p)}(b) := \phi_f(\pi_{p,1}(b)) - \frac{1}{\alpha_p}\cdot\phi_f(\pi_{p,2}(b))$$

where α_p is the unit root of the p-th Hecke polynomial of f and $b\in B^\times\backslash\widehat{B}^\times/\widehat{R_0(p)}^\times$.

Remark 2.5.
(1) The p-stabilized automorphic form $\phi_f^{(\alpha_p)}$ is a Hecke eigenform of level pN^+ and of discriminant N^-. All prime-to-p Hecke eigenvalues of $\phi_f^{(\alpha_p)}$ coincide with those of ϕ_f and the U_p-eigenvalue of $\phi_f^{(\alpha_p)}$ is α_p.
(2) Degeneracy maps at $\ell\neq p$ will be used significantly in §5. See Definition 5.3. Those degeneracy maps are essential to strip out Euler factors at ℓ.

Note that automorphic forms can be regarded as functions on a finite graph:

$$\phi_f : B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \simeq \Gamma \backslash \mathcal{V}(BT_p) \to \mathcal{O}$$

Degeneracy maps can be interpreted in terms of the graph. Two automorphic forms $\phi_{f,s}$ and $\phi_{f,t}$ of level pN^+ and discriminant N^- are defined by

$$\phi_{f,s}, \phi_{f,t} : B^\times \backslash \widehat{B}^\times / \widehat{R_0(p)}^\times \simeq \Gamma \backslash \overrightarrow{\mathcal{E}}(BT_p) \to \mathcal{O}$$

by $\phi_{f,s}(e) := \phi_f(s(e))$ and $\phi_{f,t}(e) := \phi_f(t(e))$ where e is an oriented edge, $s(e)$ is the source of the edge, and $t(e)$ is the target of the edge. The p-stabilization can be written as follows:

$$\begin{aligned} \phi_f^{(\alpha_p)}(e) &:= \phi_{f,t}(e) - \frac{1}{\alpha_p} \cdot \phi_{f,s}(e) \\ &= \phi_f(t(e)) - \frac{1}{\alpha_p} \cdot \phi_f(s(e)) \end{aligned}$$

where $e \in \Gamma \backslash \overrightarrow{\mathcal{E}}(BT_p)$.

2.3. **Construction of p-adic L-functions.** We sketch the construction of anticyclotomic p-adic L-functions of modular forms. We closely follow [BD05] with correction consulting [BD01] and [BDIS02]. In this section, modular form f is not necessarily a newform; it is just an eigenform which is N^--new.

The triple (f, K, p) determines the element $L_p(K_\infty, f) \in \Lambda$, which interpolates the special values of the Rankin L-function $L(f/K, \chi, 1)$ for all finite order characters χ of Γ_∞ of p-power conductors.

Remark 2.6. The construction of measures involves various choices and is given only up to multiplication by an element of the Galois group. However, all the ambiguities will be removed in §2.3.7.

2.3.1. *The Galois group.* Let $\mathcal{O}_K$ be the ring of integers of K and $\mathcal{O}_K[\frac{1}{p}]$ be the maximal $\mathbb{Z}[\frac{1}{p}]$-order in K. Let

$$\widetilde{G}_\infty := K^\times \backslash \widehat{K}^\times / (\widehat{\mathbb{Q}}^\times \cdot \prod_{\ell \neq p} \mathcal{O}_K[\frac{1}{p}]_\ell^\times).$$

Using global class field theory, this group is the Galois group of the union of the ring class fields of K of all p-power conductors over K, which contains the anticyclotomic $\mathbb{Z}_p$-extension of K.

2.3.2. *Optimal embeddings and the "partial" Galois action.*

Choice 2.7. *Fix an oriented optimal embedding $\Psi : K \hookrightarrow B$ such that $\Psi(K) \cap R[\frac{1}{p}] = \Psi(\mathcal{O}_K[\frac{1}{p}])$ where R is an oriented Eichler order of level N^+.*

The decomposition $N = N^+ \cdot N^-$ ensures the existence of such an embedding Ψ ([Vig80, §3 of chapitre II and §5.C of chapitre III]). The oriented optimal embedding induces the p-adic embedding $\Psi_p : K_p^\times/\mathbb{Q}_p^\times \hookrightarrow B_p^\times/\mathbb{Q}_p^\times = \mathrm{PGL}_2(\mathbb{Q}_p)$. Hence, this embedding yields the action of $K_p^\times/\mathbb{Q}_p^\times$ on the Bruhat-Tits tree BT_p **via left translation**.

We describe the structure of $\widetilde{G}_\infty$:

$$\begin{tikzcd} & K_p^\times/\mathbb{Q}_p^\times \arrow[d] \arrow[dr] & & & \\ 1 \arrow[r] & G_\infty := K_p^\times/\mathbb{Q}_p^\times(\mathcal{O}_K[\frac{1}{p}])^\times \arrow[r] & \widetilde{G}_\infty \arrow[r] \arrow[d] & \mathrm{Cl}(\mathcal{O}_K[\frac{1}{p}]) \arrow[r] & 1 \\ & & \Gamma_\infty & & \end{tikzcd}$$

Remark 2.8. The class group $\mathrm{Cl}(\mathcal{O}_K[\frac{1}{p}])$ permutes oriented optimal embeddings transitively. For $\delta \in \mathrm{Cl}(\mathcal{O}_K[\frac{1}{p}])$, the permuted optimal embedding Ψ^δ by δ is explicitly defined in [BDIS02, §2.3].

2.3.3. *Measures on $K_p^\times/\mathbb{Q}_p^\times$ and Gross points on the tree.* We define the measure on $K_p^\times/\mathbb{Q}_p^\times$ attached to f and Ψ. Consider the decreasing filtration of $K_p^\times/\mathbb{Q}_p^\times$

$$\cdots \subseteq U_{n+1} \subseteq U_n \subseteq U_{n-1} \subseteq \cdots \subseteq U_1 \subseteq U_0 \subseteq K_p^\times/\mathbb{Q}_p^\times$$

where U_0 is the maximal compact subgroup of $K_p^\times/\mathbb{Q}_p^\times$ and $U_n = (1 + p^n\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)/(1 + p^n\mathbb{Z}_p)$ for each $n \geq 1$.

Choice 2.9. *Choose a sequence of consecutive vertices $v_1, v_2, v_3, \cdots$ of $\mathcal{V}(BT_p)$ with the coherent orientation and without backtracking such that $\mathrm{Stab}_{\Psi_p(K_p^\times/\mathbb{Q}_p^\times)}(v_n) = U_n$ for all $n \geq 1$.*

Each vertex v_n is called **a Gross point of conductor p^n (at level 0)**. The $\mathcal{O}$-valued measure $\widetilde{\mu}_{f,\Psi}$ on $K_p^\times/\mathbb{Q}_p^\times$ is defined by the value

$$\begin{aligned}\widetilde{\mu}_{f,\Psi}(\sigma \cdot U_n) &:= \frac{1}{\alpha_p^n} \cdot \left(\phi_f(\Psi_p(\sigma) \cdot v_n) - \frac{1}{\alpha_p} \phi_f(\Psi_p(\sigma) \cdot v_{n-1}) \right) \\ &= \frac{1}{\alpha_p^n} \cdot \left(\phi_f^{(\alpha_p)}(\Psi_p(\sigma) \cdot e_n) \right)\end{aligned}$$

where $\sigma \in \left(K_p^\times/\mathbb{Q}_p^\times\right)/U_n$ and e_n is the oriented edge from v_{n-1} to v_n. Each oriented edge e_n is called **a Gross point of conductor p^n (at level 1)**. Since $\phi_f^{(\alpha_p)}$ is U_p-eigenform with p-adic unit eigenvalue α_p, the distribution relation follows immediately.

2.3.4. *Measures on G_∞.* The measure on $K_p^\times/\mathbb{Q}_p^\times$ on naturally induces measure $\mu_{f,\Psi}$ on the quotient $G_\infty = K_p^\times/(\mathbb{Q}_p^\times \mathcal{O}_K[\frac{1}{p}]^\times)$ as follows:

Let $\overline{U}_n \subseteq G_\infty$ be the image of U_n in G_∞ and $G_n := G_\infty/\overline{U}_n$. Then we define

$$\mu_{f,\Psi}(\sigma \cdot \overline{U}_n) := \widetilde{\mu}_{f,\Psi}(\widetilde{\sigma} \cdot U_n)$$

where $\sigma \in G_n$ and $\widetilde{\sigma}$ is any lift of σ to $\left(K_p^\times/\mathbb{Q}_p^\times\right)/U_n$. The value is independent of the choice of this lifting because ϕ_f is $R[\frac{1}{p}]^\times$-invariant and $\Psi(\mathcal{O}_K[\frac{1}{p}]) \subseteq R[\frac{1}{p}]$.

2.3.5. *Measures on $\widetilde{G}_\infty$ and theta elements.* We extend the measure on G_∞ to $\widetilde{G}_\infty$.

Choice 2.10. *We choose a lifting of $\mathrm{Cl}(\mathcal{O}_K[1/p])$ to $\widetilde{G}_\infty$. For $\delta \in \mathrm{Cl}(\mathcal{O}_K[1/p])$, write $\widetilde{\delta}$ for the lifting of δ to $\widetilde{G}_\infty$.*

Consider a coset decomposition

$$\widetilde{G}_\infty = \coprod_\delta \widetilde{\delta} G_\infty$$

where δ runs over $\mathrm{Cl}(\mathcal{O}_K[1/p])$. For $\widetilde{\sigma} \in \widetilde{G}_\infty$, write $\widetilde{\sigma} = \widetilde{\delta} \cdot \sigma$ with $\sigma \in G_\infty$ using Choice 2.10. Then we define **the measure $\mu_{f,K}$ on $\widetilde{G}_\infty$** by

$$\mu_{f,K}(\widetilde{\sigma}\overline{U}_n) := \mu_{f,\Psi^{\delta^{-1}}}(\sigma\overline{U}_n).$$

See also [BD01, §4.1.Step 5]. Note that this extended measure is defined up to multiplication by an element of G_∞ due to Choice 2.10.

Definition 2.11. (Theta elements) Let $\widetilde{G}_n := \widetilde{G}_\infty/\overline{U}_n$. Let

$$\widetilde{\theta}_{f,\Psi,n} := \sum_{\sigma \in G_n} \mu_{f,\Psi}(\sigma\overline{U}_n) \cdot \sigma^{-1} \in \mathcal{O}[G_n]$$

and

$$\begin{aligned}\widetilde{\theta}_{f,n} &:= \sum_{\delta \in \mathrm{Cl}(\mathcal{O}_K[1/p])} \left(\widetilde{\theta}_{f,\Psi^{\delta^{-1}},n} \cdot \widetilde{\delta}^{-1} \right) \\ &= \sum_{\sigma \in G_n} \sum_{\delta \in \mathrm{Cl}(\mathcal{O}_K[1/p])} \mu_{f,K}(\sigma\widetilde{\delta}\overline{U}_n)\sigma^{-1} \cdot \widetilde{\delta}^{-1} \\ &= \sum_{\widetilde{\sigma} \in \widetilde{G}_n} \mu_{f,K}(\widetilde{\sigma}\overline{U}_n)\widetilde{\sigma}^{-1} \in \mathcal{O}[\widetilde{G}_n].\end{aligned}$$

Using the distribution relation, we have

$$\widetilde{\theta}_f := \left(\varprojlim_n \widetilde{\theta}_{f,n} \right) \in \mathcal{O}[\![\widetilde{G}_\infty]\!]$$

where the inverse limit is taken under natural projection. Define the **n-th layer of the theta element $\theta_{f,n}$ of f** and the **theta element θ_f of f** to be the images of $\widetilde{\theta}_{f,n}$ and $\widetilde{\theta}_f$ in $\mathcal{O}[\Gamma_n]$ and $\mathcal{O}[\![\Gamma_\infty]\!]$, respectively.

Remark 2.12. (on choices and ambiguities)

(1) The element $\widetilde{\theta}_f$ depends on Choice 2.7 only up to sign and up to multiplication by an element of $\widetilde{G}_\infty$. See [BD01, Lemma 4.5].
(2) Choice 2.9 does not harm the definition since $\widetilde{G}_\infty$ acts on Gross points simply transitively. See [BD01, Lemma 4.3].
(3) In [BD05, §1.2], the construction argument tacitly assumes $p \nmid \#Cl(\mathcal{O}_K[\frac{1}{p}])$ to avoid Choice 2.10.

2.3.6. *"Galois action" on measures with the adelic map.* Let $\widehat{R}^{(p)} := R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)}$ where $\widehat{\mathbb{Z}}^{(p)}$ is the ring of finite prime-to-p integral adeles. The chosen optimal embedding Ψ also induces the adelic map $\widehat{\Psi} : \widetilde{G}_\infty \to B^\times \backslash \widehat{B}^\times / \widehat{R}^{(p),\times}$ and the "action" of $\widetilde{G}_\infty$ on $B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$ as follows.

Consider the composition of two maps

$$\eta \circ \widehat{\Psi} : \widetilde{G}_\infty = K^\times \backslash \widehat{K}^\times / (\widehat{\mathbb{Q}}^\times \cdot \textstyle\prod_{\ell \neq p} \mathcal{O}_K[\frac{1}{p}]^\times_\ell) \xrightarrow{\widehat{\Psi}} B^\times \backslash \widehat{B}^\times / \widehat{R}^{(p),\times} \xrightarrow{\eta} B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \tag{2}$$

where the second map η is the natural quotient. This yields the "Galois action" of $\widetilde{G}_\infty$ via left translation.

Let $\gamma \in \widetilde{G}_\infty$ and write $\gamma = \widetilde{\delta} \cdot r$ using Choice 2.10 with $r \in G_\infty$. Then we have the equation ([BD01, §4.1.Step 5]):

$$\eta \circ \widehat{\Psi}(\gamma) = \eta' \circ \Psi_p^{\delta^{-1}}(r) \tag{3}$$

up to multiplication by an element of G_∞ where $\eta' : B_p^\times / \mathbb{Q}_p^\times \to R[1/p]^\times \backslash B_p^\times / \mathbb{Q}_p^\times$ is the natural quotient map and the identity uses the identification (1). Equation (3) shows the explicit relation between Ψ_p and $\widehat{\Psi}$.

Let $\sigma \in \widetilde{G}_\infty$ and e_n be a Gross point of conductor p^n at level 1 and U_n be the stabilizer of e_n. Then we define the action of $\widetilde{G}_\infty$ on $\mu_{f,K}$ as follows.

$$(\mu_{f,K} \mid_\sigma)(\overline{U}_n) := \mu_{f,K}(\sigma \overline{U}_n) = \frac{1}{\alpha_p^n} \cdot \phi_f^{(\alpha_p)}(\eta \circ \widehat{\Psi}(\sigma) e_n). \tag{4}$$

2.3.7. *Cleaning up all the choices and p-adic L-functions.* Consider the natural involution on the Iwasawa algebra $\Lambda = \mathcal{O}[\![\Gamma_\infty]\!]$:

$$\begin{aligned} (-)^* : \mathcal{O}[\![\Gamma_\infty]\!] &\to \mathcal{O}[\![\Gamma_\infty]\!] \\ \theta = \sum a_\sigma \sigma &\mapsto \theta^* = \sum a_\sigma \sigma^{-1}. \end{aligned}$$

Definition 2.13. (The anticyclotomic p-adic Rankin L-functions) The **anticyclotomic p-adic Rankin L-function attached to f and K** is the element $L_p(K_\infty, f) \in \mathcal{O}[\![\Gamma_\infty]\!]$ defined by

$$L_p(K_\infty, f) := \theta_f \theta_f^*.$$

Proposition 2.14. *The element $L_p(K_\infty, f)$ is a well-defined element in Λ.*

Proof. Straightforward. □

For the interpolation formula, see [CH16, Theorem A].

3. Congruences and mod p multiplicity one

In this section, we provide necessary tools to prove the first part of the main theorem (Theorem 6.1). We focus on the modular forms whose residual Hecke eigensystems are *completely* the same.

3.1. **Congruences of automorphic forms.** We first consider a concept of stronger congruences.

Definition 3.1. (Fully congruent forms) Let f and g be eigenforms. We say f and g are **fully congruent** if $a_\ell(f) \equiv a_\ell(g) \pmod{\mathfrak{p}}$ for *all* primes ℓ where $\mathfrak{p}$ is the prime ideal of $\mathcal{O}$, a finite extension of the compositum of the Hecke fields of f and g.

Because of the construction of the p-adic L-functions, the following theorem directly implies the congruences between p-adic L-functions (Theorem 6.2.(1)).

Theorem 3.2. (Congruences between automorphic forms) *Assume $(\overline{\rho}, N^-)$ satisfies condition CR. Let $f, g \in S_2(\Gamma_0(N))^{N^-\text{-new}}$ such that $\overline{\rho} \simeq \overline{\rho}_f \simeq \overline{\rho}_g$ and $N^- = N(\rho_f)^- = N(\rho_g)^-$. If f and g are fully congruent, we have the congruence between the corresponding automorphic forms*

$$\phi_f \equiv u \cdot \phi_g \pmod{\mathfrak{p}}$$

where $\mathfrak{p}$ is the prime ideal of $\mathcal{O}$ (which is compatible with the embedding ι_p) and $u \in \mathcal{O}^\times$.

Note that automorphic forms are well-defined only up to a p-adic unit due to the normalization. The proof will be given in §3.4. We review the necessary material for the proof in next two sections.

3.2. **Ingredients from arithmetic of Shimura curves.** In the classical case, mod p multiplicity one can be proved using the q-expansion principle in characteristic p ([Wil95, §2.1]), but the q-expansion principle is not available in the quaternionic setting because Gross and Shimura curves have no cusps. Nevertheless, we have the following multiplicity one result for Shimura curves under condition CR.

Let $X^{N^-/r}(K_0(rN^+))$ be the Shimura curve of level rN^+ and discriminant N^-/r over $\mathbb{F}_{r^2}$ with a prime divisor $r \mid N^-$ and $\mathcal{X}_r(rN^+, N^-/r)$ be the character group of the maximal torus of the bad reduction of the Néron model of the Jacobian of $X^{N^-/r}(K_0(rN^+))$.

Theorem 3.3. ([PW11, Theorem 6.2], [Hel07]) *Suppose that $(\overline{\rho}_f, N^-)$ satisfies condition CR. Then the completion of the character group $(\mathcal{X}_r(rN^+, N^-/r) \otimes_{\mathbb{Z}} \mathcal{O})_{\mathfrak{m}_f}$ at $\mathfrak{m}_f$ is free of rank one over the completed Hecke algebra $(\mathbb{T}^{N^-}(K_0(N^+))_{\mathcal{O}})_{\mathfrak{m}_f}$.*

Remark 3.4. Condition CR is optimal for this multiplicity one theorem. See [Rib90a] for a counterexample without condition CR.

Using Ribet's argument of changing indefinite and definite quaternion algebras, we transplant the above multiplicity one result to the definite quaternionic setting.

Let $\mathbb{Z}[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times]$ be the free abelian group generated by $B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$ and $\mathbb{Z}[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times]^0$ be its subgroup which is the kernel of the augmentation map $\mathbb{Z}[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times] \to \mathbb{Z}$ defined by $\sum_b a_b \cdot b \mapsto \sum_b a_b$ where $b \in B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$ and $a_b \in \mathbb{Z}$.

Theorem 3.5. ([PW11, Proposition 6.5], [Rib90b]) *For each prime divisor $r \mid N^-$, there is a canonical $\mathbb{T}^{N^-}(K_0(N^+))_{\mathcal{O}}$-equivariant isomorphism*

$$\mathbb{Z}[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times]^0 \otimes_{\mathbb{Z}} \mathcal{O} \simeq \mathcal{X}_r(rN^+, N^-/r) \otimes_{\mathbb{Z}} \mathcal{O}$$

and this isomorphism takes the intersection pairing on $\mathbb{Z}[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times]^0$ to the monodromy pairing on $\mathcal{X}_r(rN^+, N^-/r)$.

3.3. **Mod p multiplicity one.** With Theorem 3.3 and 3.5, we prove the following "mod p multiplicity one" result for automorphic forms on a definite quaternion algebra.

Lemma 3.6. (Mod ϖ multiplicity one for automorphic forms) *Assume that $(\overline{\rho}_f, N^-)$ satisfies condition CR, and let $\mathfrak{m}_f$ be the maximal ideal of the Hecke algebra $\mathbb{T}^{N^-}(K_0(N^+))_{\mathcal{O}}$ corresponding to $\overline{\rho}_f$. Then*

$$S_2^{N^-}(K_0(N^+), \mathbb{F})_{\mathfrak{m}_f}[\mathfrak{m}_f] \simeq \mathbb{F}.$$

Proof. First, we simplify notation:

- $\mathbb{T} = \mathbb{T}^{N^-}(K_0(N^+))_{\mathcal{O}}$, the Hecke algebra over $\mathcal{O}$
- $\mathfrak{p} = \ker(\pi_f : \mathbb{T} \to \mathcal{O}; T_\ell \mapsto a_\ell(f)) \subseteq \mathbb{T}$, the height 1 prime corresponding to an automorphic eigenform ϕ_f
- $\mathfrak{m} = \mathfrak{m}_f =$ the maximal ideal of $\mathbb{T}$ containing $\mathfrak{p}$ and ϖ
- $S_2(\mathcal{O}) = S_2^{N^-}(K_0(N^+), \mathcal{O})$

- $S_2(\mathbb{F}) = S_2^{N^-}(K_0(N^+), \mathbb{F})$.

Consider a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{\mathfrak{m}}/\mathfrak{p}\mathbb{T}_{\mathfrak{m}}, \mathcal{O}) \simeq \mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{\mathfrak{m}}, \mathcal{O})[\mathfrak{p}].$$

Since the $\mathfrak{m}$-adic completion of the character group $(\mathcal{X}_r(rN^+, N^-/r) \otimes_{\mathbb{Z}} \mathcal{O})_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}_{\mathfrak{m}}$ (Theorem 3.3), we have

$$\mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{\mathfrak{m}}, \mathcal{O})[\mathfrak{p}] \simeq \mathrm{Hom}_{\mathcal{O}}((\mathcal{X}_r(rN^+, N^-/r) \otimes_{\mathbb{Z}} \mathcal{O})_{\mathfrak{m}}, \mathcal{O})[\mathfrak{p}].$$

By Theorem 3.5,

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{X}_r(rN^+, N^-/r) \otimes_{\mathbb{Z}} \mathcal{O})_{\mathfrak{m}}, \mathcal{O})[\mathfrak{p}] \simeq \mathrm{Hom}_{\mathcal{O}}((\mathbb{Z}[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times]^0 \otimes_{\mathbb{Z}} \mathcal{O})_{\mathfrak{m}}, \mathcal{O})[\mathfrak{p}].$$

By the (twisted) Hecke-equivariant self-duality of automorphic forms (e.g. [BD05, §1.1], [SW99, §3.2]), we have

$$\mathrm{Hom}_{\mathcal{O}}((\mathbb{Z}[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times]^0 \otimes_{\mathbb{Z}} \mathcal{O})_{\mathfrak{m}}, \mathcal{O}) \simeq \mathrm{Hom}_{\mathcal{O}}(\mathbb{Z}[B^\times \backslash \widehat{B}^\times / \widehat{R}^\times]^0 \otimes_{\mathbb{Z}} \mathcal{O}, \mathcal{O})_{\mathfrak{m}} = S_2(\mathcal{O})_{\mathfrak{m}}.$$

Thus, we have

$$S_2(\mathcal{O})_{\mathfrak{m}}[\mathfrak{p}] \simeq \mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{\mathfrak{m}}/\mathfrak{p}\mathbb{T}_{\mathfrak{m}}, \mathcal{O}).$$

Since $\mathbb{T}_{\mathfrak{m}}/\mathfrak{p}\mathbb{T}_{\mathfrak{m}} \simeq \mathcal{O}$, its $\mathcal{O}$-dual $\mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{\mathfrak{m}}/\mathfrak{p}\mathbb{T}_{\mathfrak{m}}, \mathcal{O})$ is also free of rank one over $\mathcal{O}$. Taking mod ϖ reduction of all the forms in LHS and all the homomorphisms in RHS, we have

$$S_2(\mathbb{F})_{\mathfrak{m}}[\mathfrak{p}] \simeq \mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{\mathfrak{m}}/\mathfrak{p}\mathbb{T}_{\mathfrak{m}}, \mathbb{F}).$$

Also, we automatically have $S_2(\mathbb{F})_{\mathfrak{m}}[\mathfrak{p}] = S_2(\mathbb{F})_{\mathfrak{m}}[\mathfrak{m}]$ and $\mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{\mathfrak{m}}/\mathfrak{p}_f\mathbb{T}_{\mathfrak{m}}, \mathbb{F}) = \mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{\mathfrak{m}}/\mathfrak{m}\mathbb{T}_{\mathfrak{m}}, \mathbb{F}) = \mathrm{Hom}_{\mathbb{F}}(\mathbb{T}_{\mathfrak{m}}/\mathfrak{m}\mathbb{T}_{\mathfrak{m}}, \mathbb{F})$. Since $\mathbb{T}_{\mathfrak{m}}/\mathfrak{m}\mathbb{T}_{\mathfrak{m}} \simeq \mathbb{F}$, we get the conclusion $S_2(\mathbb{F})_{\mathfrak{m}}[\mathfrak{m}] \simeq \mathbb{F}$. □

3.4. **Proof of Theorem 3.2.** We give a proof of Theorem 3.2.

Proof of Theorem 3.2. Let f and g be fully congruent classical forms. Then the corresponding height 1 primes $\mathfrak{p}_f$ and $\mathfrak{p}_g$ are contained the same maximal ideal $\mathfrak{m}$. By Lemma 3.6, we know $S_2^{N^-}(K_0(N^+), \mathbb{F})_{\mathfrak{m}}[\mathfrak{m}]$ is one-dimensional over $\mathbb{F}$ where $\mathbb{F}$ is a finite extension of the compositum of residue fields of the Hecke fields of f and g. This implies that one automorphic form, say ϕ_f, is a constant multiple of the other form ϕ_g. Since both are nonzero modulo ϖ due to the normalization, the constant is a unit. Thus, we have the congruence

$$\phi_f \equiv u \cdot \phi_g \pmod{\varpi}$$

where u is a unit in $\mathcal{O}$. □

4. Local pieces of the main conjecture

In this section, we introduce Iwasawa-theoretic Euler factors and focus on "the Iwasawa main conjecture *for one Euler factor*." More precisely, we state and prove an explicit relation between the Euler factor at a prime ℓ which splits in K and the corresponding algebraic invariant, the local cohomology group at ℓ.

Let $\ell(\neq p)$ be a prime which splits in K. Write $\ell = \mathfrak{l}\bar{\mathfrak{l}}$ in K.

4.1. **Cohomological and cyclotomic Iwasawa-theoretic Euler factors.**

Definition 4.1. (Cohomological Euler factors: [GV00]) We define the **cohomological Euler factor** $E_\ell(f, X)$ **at** $\ell(\neq p)$ of the L-function attached to $\mathfrak{p}$-adic modular Galois representation V_f by

$$E_\ell(f, T) := \det((1 - \mathrm{Fr}_\ell \cdot T) \mid_{(V_f)_{I_\ell}}) \in \mathcal{O}[T]$$

where Fr_ℓ is the arithmetic Frobenius at ℓ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Convention 4.2. In the definition, we take the maximal unramified *quotient* of V_f at ℓ following [Gre89, (5)] and [GV00, Proposition 2.4].

Let $\Gamma^{\mathrm{cyc}}_\infty = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ be the Galois group of the cyclotomic $\mathbb{Z}_p$-extension over $\mathbb{Q}$ and $\Lambda^{\mathrm{cyc}} = \mathcal{O}[\![\Gamma^{\mathrm{cyc}}_\infty]\!]$ be the cyclotomic Iwasasa algebra.

Definition 4.3. (Cyclotomic Iwasawa-theoretic Euler factors) The **Iwasawa-theoretic Euler factor** $\mathcal{E}_\ell(\mathbb{Q}_\infty, f)$ **at** ℓ is defined by

$$\mathcal{E}_\ell(\mathbb{Q}_\infty, f) := E_\ell(f, \ell^{-1}\gamma_\ell) \in \Lambda^{\mathrm{cyc}}$$

where γ_ℓ is the arithmetic Frobenius at ℓ in $\Gamma^{\mathrm{cyc}}_\infty$.

Theorem 4.4. ([GV00, Proposition 2.4])

$$\mathrm{char}_{\Lambda^{\mathrm{cyc}}}\left(\oplus_{v|\ell}\mathrm{H}^1\left(\mathbb{Q}_{\infty,v},A_f\right)^\vee\right)=\mathcal{E}_\ell(\mathbb{Q}_\infty,f)\cdot\Lambda^{\mathrm{cyc}}.$$

Remark 4.5. The Iwasawa-theoretic Euler factor $\mathcal{E}_\ell(\mathbb{Q}_\infty,f)$ satisfies the interpolation property: $\chi(\mathcal{E}_\ell(\mathbb{Q}_\infty,f))=E_\ell(f,\chi(\ell)\ell^{-1})$ where $\chi:\Gamma^{\mathrm{cyc}}_\infty\to\overline{\mathbb{Q}}^\times$. Note that we identify γ_ℓ and ℓ via class field theory. See [GV00, Proposition 2.4] for the Iwasawa-theoretic properties of the Euler factor.

4.2. **Anticyclotomic Iwasawa-theoretic Euler factors.** The Euler factor at ℓ of the Rankin-Selberg L-function attached to f and K twisted by an anticyclotomic character χ is given by

- for ℓ with $\ell\nmid N$,

$$\begin{aligned}E_\ell(f/K,\chi,s)&=(1-\alpha_\ell(f)\chi(\mathfrak{l})\ell^{-s})(1-\alpha_\ell(f)\chi(\bar{\mathfrak{l}})\ell^{-s})(1-\beta_\ell(f)\chi(\mathfrak{l})\ell^{-s})(1-\beta_\ell(f)\chi(\bar{\mathfrak{l}})\ell^{-s})\\&=E_{\mathfrak{l}}(f/K,\chi,s)\cdot E_{\bar{\mathfrak{l}}}(f/K,\chi,s)\end{aligned}$$

  where

$$\begin{aligned}E_{\mathfrak{l}}(f/K,\chi,s)&=(1-\alpha_\ell(f)\chi(\mathfrak{l})\ell^{-s})(1-\beta_\ell(f)\chi(\mathfrak{l})\ell^{-s})\\E_{\bar{\mathfrak{l}}}(f/K,\chi,s)&=(1-\alpha_\ell(f)\chi(\bar{\mathfrak{l}})\ell^{-s})(1-\beta_\ell(f)\chi(\bar{\mathfrak{l}})\ell^{-s}),\end{aligned}$$

- for ℓ with $\ell\|N$,

$$\begin{aligned}E_\ell(f/K,\chi,s)&=(1-a_\ell(f)\chi(\mathfrak{l})\ell^{-s})(1-a_\ell(f)\chi(\bar{\mathfrak{l}})\ell^{-s})\\&=E_{\mathfrak{l}}(f/K,\chi,s)\cdot E_{\bar{\mathfrak{l}}}(f/K,\chi,s)\end{aligned}$$

  where

$$\begin{aligned}E_{\mathfrak{l}}(f/K,\chi,s)&=(1-a_\ell(f)\chi(\mathfrak{l})\ell^{-s})\\E_{\bar{\mathfrak{l}}}(f/K,\chi,s)&=(1-a_\ell(f)\chi(\bar{\mathfrak{l}})\ell^{-s}),\end{aligned}$$

- for ℓ with $\ell^2\mid N$,

$$E_\ell(f/K,\chi,s)=1.$$

Definition 4.6. (Iwasawa-theoretic Euler factors at primes which split in K) Let ℓ be a prime which splits in K and N be the level of f. The **Iwasawa-theoretic Euler factor at ℓ of the Rankin L-function in** Λ is defined by

$$\mathcal{E}_\ell(K_\infty,f):=\begin{cases}(1-\alpha_\ell(f)\ell^{-1}\gamma_{\mathfrak{l}})(1-\alpha_\ell(f)\ell^{-1}\gamma_{\bar{\mathfrak{l}}})(1-\beta_\ell(f)\ell^{-1}\gamma_{\mathfrak{l}})(1-\beta_\ell(f)\ell^{-1}\gamma_{\bar{\mathfrak{l}}}) & \text{if } \ell\nmid N\\(1-a_\ell(f)\ell^{-1}\gamma_{\mathfrak{l}})(1-a_\ell(f)\ell^{-1}\gamma_{\bar{\mathfrak{l}}}) & \text{if } \ell\|N\\1 & \text{if } \ell^2\mid N\end{cases}$$

where $\ell=\mathfrak{l}\bar{\mathfrak{l}}$ in K, and $\gamma_{\mathfrak{l}}$ and $\gamma_{\bar{\mathfrak{l}}}$ are the arithmetic Frobenii at $\mathfrak{l}$ and $\bar{\mathfrak{l}}$, respectively, in Γ_∞.

Similarly, we have

$$\mathcal{E}_{\mathfrak{l}}(K_\infty,f)=\begin{cases}(1-\alpha_\ell(f)\ell^{-1}\gamma_{\mathfrak{l}})(1-\beta_\ell(f)\ell^{-1}\gamma_{\mathfrak{l}}) & \text{if } \ell\nmid N\\(1-a_\ell(f)\ell^{-1}\gamma_{\mathfrak{l}}) & \text{if } \ell\|N\\1 & \text{if } \ell^2\mid N\end{cases},$$

and

$$\mathcal{E}_{\bar{\mathfrak{l}}}(K_\infty,f)=\begin{cases}(1-\alpha_\ell(f)\ell^{-1}\gamma_{\bar{\mathfrak{l}}})(1-\beta_\ell(f)\ell^{-1}\gamma_{\bar{\mathfrak{l}}}) & \text{if } \ell\nmid N\\(1-a_\ell(f)\ell^{-1}\gamma_{\bar{\mathfrak{l}}}) & \text{if } \ell\|N\\1 & \text{if } \ell^2\mid N\end{cases}.$$

Theorem 4.7. (Anticyclotomic analogue of Theorem 4.4) *Let K be an imaginary quadratic field and ℓ be a prime which splits in $\mathfrak{l}\bar{\mathfrak{l}}$ in K. Then*

$$\mathrm{char}_\Lambda\left(\oplus_{v|\ell}\mathrm{H}^1\left(G_{K_{\infty,v}},A_f\right)^\vee\right)=\mathcal{E}_\ell(K_\infty,f)\cdot\Lambda$$

where v runs over all primes of K_∞ dividing ℓ.

Proof. Since ℓ splits in K, we have $K_\ell = K_{\mathfrak{l}} \oplus K_{\bar{\mathfrak{l}}} \simeq \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$. This implies that $K_{\infty,u} \simeq K_{\infty,v} \simeq \mathbb{Q}_{\infty,w}$ where u, v, or w is a prime of K_∞, K_∞, or $\mathbb{Q}_\infty$ lying above $\mathfrak{l}$, $\bar{\mathfrak{l}}$, or ℓ, respectively.

Note that first two fields $K_{\infty,u}$ and $K_{\infty,v}$ are the completions of the anticyclotomic $\mathbb{Z}_p$-extension K_∞ of K but $\mathbb{Q}_{\infty,w}$ is the completion of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$. Thus, if implies that

$$\begin{aligned}\oplus_{v|\ell}\mathrm{H}^1(G_{K_{\infty,v}}, A_f) &= (\oplus_{v|\mathfrak{l}}\mathrm{H}^1(K_{\infty,v}, A_f)) \oplus (\oplus_{v|\bar{\mathfrak{l}}}\mathrm{H}^1(K_{\infty,v}, A_f)) \\ &= (\oplus_{v|\ell}\mathrm{H}^1(\mathbb{Q}_{\infty,v}, A_f)) \oplus (\oplus_{v|\ell}\mathrm{H}^1(\mathbb{Q}_{\infty,v}, A_f)).\end{aligned}$$

We also obtain a similar decomposition and identification for Euler factors:

$$\mathcal{E}_\ell(K_\infty, f) = \mathcal{E}_{\mathfrak{l}}(K_\infty, f) \cdot \mathcal{E}_{\bar{\mathfrak{l}}}(K_\infty, f)$$

and

$$\mathcal{E}_{\mathfrak{l}}(K_\infty, f) = \mathcal{E}_{\bar{\mathfrak{l}}}(K_\infty, f) = \mathcal{E}_\ell(\mathbb{Q}_\infty, f).$$

By Theorem 4.4, we have the conclusion. □

4.3. **Congruences and an algebraic interpretation of Euler factors.** We record an anticyclotomic version of an argument in [EPW06] without any serious modification. We use this result for application of the main theorem to the main conjecture (Corollary 6.7).

Let ℓ be a rational prime which splits in K. Let

$$\delta_v(f) := \dim_{\mathbb{F}}\left(A_f^{G_{K_{\infty,v}}}/\varpi A_f^{G_{K_{\infty,v}}}\right)$$

where v is a prime of K_∞ lying over ℓ. Write $\delta_\ell(f) := \sum_{v|\ell}\delta_v(f)$, $\delta_{\mathfrak{l}}(f) := \sum_{v|\mathfrak{l}}\delta_v(f)$, and $\delta_{\bar{\mathfrak{l}}}(f) := \sum_{v|\bar{\mathfrak{l}}}\delta_v(f)$. Then we have $\delta_\ell(f) = \delta_{\mathfrak{l}}(f) + \delta_{\bar{\mathfrak{l}}}(f)$.

Theorem 4.8. ([EPW06, Lemma 5.1.5]) *Let f and g be fully congruent eigenforms mod ϖ. Then, for any prime $\ell \neq p$ which splits in K,*

$$\sum_{v|\ell}(\delta_v(f) - \delta_v(g)) = \lambda(\mathcal{E}_\ell(K_\infty, g)) - \lambda(\mathcal{E}_\ell(K_\infty, f))$$

where the sum runs over all primes v of K_∞ lying over ℓ.

Proof. Identical with the cyclotomic case. □

5. Ihara's lemma and stripping out Euler factors

In general, the same residual modular representation does not imply the *full* congruence of two congruent eigenforms in the sense of Definition 3.1. Finitely many residual Hecke eigenvalues may differ. In order to remove this difference, we raise the levels of two congruent modular forms to make them fully congruent. In the aspect of L-functions, it corresponds to removing Euler factors of L-functions of two eigenforms at primes dividing their levels.

The goal of this section is to prove the following theorem.

Theorem 5.1. (Removing the Euler factor at a splitting prime) *Let $f = \sum_{n\geq 1} a_n q^n$ be an eigenform of level $\Gamma_0(N)$ which is N^--new. Assume that $(\bar{\rho}_f, N^-)$ satisfies condition CR. Let ℓ be a rational prime which splits in K and is prime to p. Let $f^{(\ell)} = \sum_{(n,\ell)=1} a_n q^n$ be the eigenform of level $\Gamma_0(N\ell^2)$ (if ℓ is prime to N) or $\Gamma_0(N\ell)$ (if ℓ exactly divides N) whose prime to ℓ Hecke eigenvalues coincide with those of f and U_ℓ-eigenvalue is 0. Then the anticyclotomic p-adic L-functions of f and $f^{(\ell)}$ have the following property:*

$$L_p(K_\infty, f^{(\ell)}) = u \cdot \mathcal{E}_\ell(K_\infty, f) \cdot L_p(K_\infty, f)$$

where $u \in \mathcal{O}^\times$.

Remark 5.2. (on the interpolation formula and inert primes)

(1) Note that this formula is compatible with the interpolation formula although p-adic L-functions do *not* admit an Euler product decomposition.

(2) Removing an Euler factor at an inert prime is a completely different situation. For a prime ℓ dividing N^-, the Euler factors of two congruent forms exactly coincide, so we do not have to remove them. For a prime not dividing N^- and inert in K, the context is completely different since f and $f^{(\ell)}$ have *different* signs of functional equations. This phenomena can be explained in the context of Jochnowitz congruences [BD99], comparing Rankin L-values with derivatives of Rankin L-values. The corresponding phenomena in the algebraic side is observed in [PW11, Lemma 3.4]. See also [PW11, Theorem 7.3 and Remark 7.4] for a more myterious behavior in this setting.

In §5.1, we reduce the proof of Theorem 5.1 to mod ϖ nonvanishing of $\phi^{(\ell)}_f$. We give the statement of mod ϖ nonvanishing of $\phi^{(\ell)}_f$ (Theorem 5.10 and 5.11) in §5.2, study the relevant Ihara's lemma, which is a crucial tool for proof, in §5.3. Finally, we give proofs of Theorems 5.10 and 5.11 in §5.4.

5.1. **Reduction.** In this section, we explain why the content of Theorem 5.1 is that *u is a unit in $\mathcal{O}$* and the reduction of the proof of this theorem to mod ϖ nonvanishing of a "ℓ-deprived" automorphic forms.

We introduce *more* degeneracy maps

$$\begin{aligned}\pi_{\ell,3} : B^\times\backslash\widehat{B}^\times/\widehat{R_0(\ell^2)}^\times &\to B^\times\backslash\widehat{B}^\times/\widehat{R_0(\ell)}^\times\\ [\cdots, b_\ell, \cdots] &\mapsto [\cdots, b_\ell, \cdots],\\ \pi_{\ell,4} : B^\times\backslash\widehat{B}^\times/\widehat{R_0(\ell^2)}^\times &\to B^\times\backslash\widehat{B}^\times/\widehat{R_0(\ell)}^\times\\ [\cdots, b_\ell, \cdots] &\mapsto [\cdots, b_\ell\left(\begin{smallmatrix}1&0\\0&\ell\end{smallmatrix}\right), \cdots]\end{aligned}$$

where the right matrix action is local at ℓ.

Definition 5.3. (ℓ-deprived automorphic forms)

(1) If ℓ is prime to N^+, then we define

$$\phi^{(\alpha_\ell)}_f := \phi_f \circ \pi_{\ell,1} - \frac{1}{\alpha_\ell}\cdot \phi_f \circ \pi_{\ell,2},$$
$$\phi^{(\beta_\ell)}_f := \phi_f \circ \pi_{\ell,1} - \frac{1}{\beta_\ell}\cdot \phi_f \circ \pi_{\ell,2},$$

and

$$\phi^{(\ell)}_f := \phi^{(\alpha_\ell)}_f \circ \pi_{\ell,3} - \frac{1}{\beta_\ell}\cdot \phi^{(\alpha_\ell)}_f \circ \pi_{\ell,4}.$$

where α_ℓ, β_ℓ are the roots of the ℓ-th Hecke polynomial $x^2 - a_\ell(f)x + \ell$ of f.

(2) If $\ell \| N^+$, then

$$\phi^{(\ell)}_f := \phi_f \circ \pi_{\ell,3} - \frac{a_\ell}{\ell}\cdot \phi_f \circ \pi_{\ell,4}$$

where a_ℓ is the U_ℓ-eigenvalue of f.

Remark 5.4. (on the properties of the above forms)

(1) $\phi^{(\alpha_\ell)}_f$ and $\phi^{(\beta_\ell)}_f$ are of level ℓN^+ under the assumption $(\ell, N^+) = 1$.
(2) $\phi^{(\ell)}_f$ is of level $\ell^2 N^+$ (if ℓ is prime to N^+) or ℓN^+ (if ℓ divides N^+ exactly).
(3) All ϕ_f, $\phi^{(\alpha_\ell)}_f$, $\phi^{(\beta_\ell)}_f$ and ϕ_f have the same prime-to-ℓ Hecke eigensystem. The U_ℓ eigenvalues of $\phi^{(\alpha_\ell)}_f$, $\phi^{(\beta_\ell)}_f$, and $\phi^{(\ell)}_f$ are α_ℓ, β_ℓ, and 0, respectively.

We fix notation which is compatible with the one given in [EPW06, §3.8]. Let

$$\begin{aligned}B^*_{\ell^2,1} &:= \pi^*_{\ell,3} \circ \pi^*_{\ell,1}\\ B^*_{\ell^2,\ell} &:= \pi^*_{\ell,4} \circ \pi^*_{\ell,1} = \pi^*_{\ell,3} \circ \pi^*_{\ell,2}\\ B^*_{\ell^2,\ell^2} &:= \pi^*_{\ell,4} \circ \pi^*_{\ell,2}\\ B^*_{\ell,1} &:= \pi^*_{\ell,1}\\ B^*_{\ell,\ell} &:= \pi^*_{\ell,2}.\end{aligned}$$

We define the "ℓ-th Euler factor removing operator" on the space of automorphic forms following [EPW06, §3.8].

Definition 5.5. Define a map

$$\epsilon(\ell) : S_2^{N^-}(N^+, \mathcal{O}) \to S_2^{N^-}(\ell^{m_\ell} N^+, \mathcal{O})$$

by

$$\epsilon(\ell) := \begin{cases} 1 & \text{if } m_\ell = 0 \ (\ell^2 \mid N^+) \\ \left(B^*_{\ell,1} - \ell^{-1} \cdot B^*_{\ell,\ell} \circ U_\ell\right) & \text{if } m_\ell = 1 \ (\ell \| N^+) \\ \left(B^*_{\ell^2,1} - \ell^{-1} \cdot B^*_{\ell^2,\ell} \circ T_\ell + \ell^{-1} \cdot B^*_{\ell^2,\ell^2}\right) & \text{if } m_\ell = 2 \ (\ell \nmid N^+). \end{cases}$$

Let $\phi_f \in S_2^{N^-}(K_0(N^+))$ be a normalized automorphic eigenform and ℓ be a prime which splits in K. Then the **fully ℓ-deprived automorphic form** $\phi_f^{(\ell)}$ is defined to be $\epsilon(\ell)\phi_f$. We investigate how $\epsilon(\ell)$ acts on measures constructed from automorphic forms in the following lemma. It also proves an anticyclotomic analogue of the latter part of [EPW06, Theorem 3.6.2] explicitly.

Lemma 5.6. *The measure attached to the fully ℓ-deprived automorphic form $\phi_f^{(\ell)}$ defines the theta element $\mathcal{E}_{\mathfrak{l}}(K_\infty, f) \cdot \theta_f$, so the corresponding p-adic L-function is $\mathcal{E}_\ell(K_\infty, f) \cdot L_p(K_\infty, f)$.*

Proof. Recalling the construction in §2.3, consider the measure constructed from $\phi_f^{(\ell)} = \epsilon(\ell)\phi_f$. Note that the optimal embedding Ψ satisfies $\Psi(K) \cap R_0(\ell^2) = \mathcal{O}_K[1/p]$ in this setting. In this calculation, we consider the $m_\ell = 2$ case, but the other case is similar. Formal computation shows that

$$\begin{aligned}
\mu^{(\ell)}_{f,\Psi}(\sigma \cdot \overline{U}_n) &:= \frac{1}{\alpha_p^n} \cdot \left(\phi_f^{(\ell)}(\Psi_p(\sigma) \cdot v_n) - \frac{1}{\alpha_p}\phi_f^{(\ell)}(\Psi_p(\sigma) \cdot v_{n-1})\right) \\
&= \frac{1}{\alpha_p^n} \cdot \epsilon(\ell) \left(\phi_f(\Psi_p(\sigma) \cdot v_n) - \frac{1}{\alpha_p}\phi_f(\Psi_p(\sigma) \cdot v_{n-1})\right) \\
&= \frac{1}{\alpha_p^n} \cdot \left(B^*_{\ell^2,1} - \ell^{-1} \cdot B^*_{\ell^2,\ell} \circ T_\ell + \ell^{-1} \cdot B^*_{\ell^2,\ell^2}\right) \circ \left(\phi_f(\Psi_p(\sigma) \cdot v_n) - \frac{1}{\alpha_p}\phi_f(\Psi_p(\sigma) \cdot v_{n-1})\right) \\
&= \epsilon(\ell)\mu_{f,\Psi}(\sigma \cdot \overline{U}_n).
\end{aligned}$$

To consider the action of $B^*_{\ell^2,1}$, $B^*_{\ell^2,\ell}$, and $B^*_{\ell^2,\ell^2}$ clearly, we regard $\Psi_p(\sigma) \cdot v_n$ and $\Psi_p(\sigma) \cdot v_{n-1}$ as elements of $B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$ as well as vertices of the Bruhat-Tits tree. Another formal computation shows that

$$\begin{aligned}
\left(B^*_{\ell^2,\ell} \circ \phi_f\right)(\Psi_p(\sigma) \cdot v_n) &:= \left(B^*_{\ell^2,1} \circ \phi_f\right)(\Psi_p(\sigma) \cdot v_n \cdot \mathrm{diag}(1,\ell)) \\
&= \left(B^*_{\ell^2,1} \circ \phi_f\right)(\Psi_p(\sigma) \cdot \mathrm{diag}(1,\ell) \cdot \mathrm{diag}(1,\ell^{-1}) \cdot v_n \cdot \mathrm{diag}(1,\ell))
\end{aligned}$$

where $\mathrm{diag}(a,b)$ is the diagonal 2×2 matrix with entries a, b and the action of diagonal matrices is local at ℓ. Under identification (1) under the strong approximation, the prime-to-p part of the Gross points is trivial, so we have $\mathrm{diag}(1,\ell^{-1}) \cdot v_n \cdot \mathrm{diag}(1,\ell) = v_n$. We continue the calculation:

$$= \left(B^*_{\ell^2,1} \circ \phi_f\right)(\Psi_p(\sigma) \cdot \mathrm{diag}(1,\ell) \cdot v_n)$$

Since the left translation by $\mathrm{diag}(1,\ell)$ corresponds to the action of arithmetic Frobenius $\gamma_{\mathfrak{l}}$ at $\mathfrak{l}$ in $\widetilde{G}_\infty$ (more precisely, its image in $\widetilde{G}_n$) where $\mathfrak{l}$ is a place of K lying over ℓ via the adelic map $\eta \circ \widehat{\Psi}$ given in (2), we have:

$$= \left(B^*_{\ell^2,1} \circ \phi_f\right)(\Psi_p(\sigma) \cdot (\eta \circ \widehat{\Psi})(\gamma_{\mathfrak{l}}) \cdot v_n).$$

Similar computation shows that

$$\begin{aligned}
\left(B^*_{\ell^2,\ell^2} \circ \phi_f\right)(\Psi_p(\sigma) \cdot v_n) &= \left(B^*_{\ell^2,1} \circ \phi_f\right)(\Psi_p(\sigma) \cdot (\eta \circ \widehat{\Psi})(\gamma_{\mathfrak{l}}^2) \cdot v_n) \\
\left(B^*_{\ell,\ell} \circ \phi_f\right)(\Psi_p(\sigma) \cdot v_n) &= \left(B^*_{\ell,1} \circ \phi_f\right)(\Psi_p(\sigma) \cdot (\eta \circ \widehat{\Psi})(\gamma_{\mathfrak{l}}) \cdot v_n).
\end{aligned}$$

Using the above computation with Equation (4), the n-th layer of theta element of $\phi_f^{(\ell)}$ can be calculated as follows:

$$\begin{aligned}
\widetilde{\theta}_{f,n}^{(\ell)} &= \sum_{\widetilde{\delta}} \sum_{\sigma \in G_n} \mu_{f,\Psi^{\delta^{-1}}}^{(\ell)}(\sigma \overline{U}_n)\sigma^{-1}\widetilde{\delta}^{-1} \\
&= \sum_{\widetilde{\sigma} \in \widetilde{G}_n} \mu_{f,K}^{(\ell)}(\widetilde{\sigma}\overline{U}_n)\widetilde{\sigma}^{-1} \\
&= \sum_{\widetilde{\sigma} \in \widetilde{G}_n} \epsilon(\ell)\mu_{f,K}(\widetilde{\sigma}\overline{U}_n)\widetilde{\sigma}^{-1} \\
&= \sum_{\widetilde{\sigma} \in \widetilde{G}_n} \left(\mu_{f,K}(\widetilde{\sigma}\overline{U}_n)\widetilde{\sigma}^{-1} - a_\ell \cdot \ell^{-1} \cdot \mu_{f,K}(\widetilde{\sigma}\gamma_{\mathfrak{l}}\overline{U}_n)\widetilde{\sigma}^{-1} + \ell^{-1} \cdot \mu_{f,K}(\widetilde{\sigma}\gamma_{\mathfrak{l}}^2\overline{U}_n)\widetilde{\sigma}^{-1}\right) \\
&= \sum_{\widetilde{\sigma} \in \widetilde{G}_n} \mu_{f,K}(\widetilde{\sigma}\overline{U}_n)\widetilde{\sigma}^{-1} - a_\ell \cdot \ell^{-1} \cdot \sum_{\widetilde{\sigma} \in \widetilde{G}_n} \mu_{f,K}(\widetilde{\sigma}\gamma_{\mathfrak{l}}\overline{U}_n)\widetilde{\sigma}^{-1} + \ell^{-1} \cdot \sum_{\widetilde{\sigma} \in \widetilde{G}_n} \mu_{f,K}(\widetilde{\sigma}\gamma_{\mathfrak{l}}^2\overline{U}_n)\widetilde{\sigma}^{-1} \\
&= \sum_{\widetilde{\sigma} \in \widetilde{G}_n} \mu_{f,K}(\widetilde{\sigma}\overline{U}_n)\widetilde{\sigma}^{-1} - a_\ell \cdot \ell^{-1} \cdot \sum_{\widetilde{\sigma} \in \widetilde{G}_n} \mu_{f,K}(\widetilde{\sigma}\overline{U}_n)\widetilde{\sigma}^{-1}\gamma_{\mathfrak{l}} + \ell^{-1} \cdot \sum_{\widetilde{\sigma} \in \widetilde{G}_n} \mu_{f,K}(\widetilde{\sigma}\overline{U}_n)\widetilde{\sigma}^{-1}\gamma_{\mathfrak{l}}^2 \\
&= \mathcal{E}_{\mathfrak{l}}(K_\infty, f) \cdot \widetilde{\theta}_{f,n}.
\end{aligned}$$

where $\mu_{f,K}^{(\ell)}(\widetilde{\sigma}\overline{U}_n) := \mu_{f,\Psi^{\delta^{-1}}}^{(\ell)}(\sigma\overline{U}_n)$ with decomposition $\widetilde{\sigma} = \widetilde{\delta} \cdot \sigma$ under Choice 2.10. □

Now we go back to the theorem.

Reduction of a proof of Theorem 5.1 to mod ϖ nonvanishing of $\phi_f^{(\ell)}$. In the statement, the theta element to construct $L_p(K_\infty, f^{(\ell)})$ is defined by the values of the *normalized* automorphic form $\phi_{f^{(\ell)}}$ of the fully ℓ-deprived classical form $f^{(\ell)}$. Lemma 5.6 shows that the values of of the fully ℓ-deprived automorphic form $\phi_f^{(\ell)}$ defines $\mathcal{E}_{\mathfrak{l}}(K_\infty, f) \cdot \theta_f$, so we have $\mathcal{E}_\ell(K_\infty, f) \cdot L_p(K_\infty, f)$. Now we compare the Hecke eigensystems of $\phi_{f^{(\ell)}}$ and $\phi_f^{(\ell)}$. By definition, the mod ϖ prime-to-ℓ Hecke eigensystems of $\phi_{f^{(\ell)}}$ and $\phi_f^{(\ell)}$ coincide. It suffices to show that the U_ℓ-eigenvalue of $\phi_f^{(\ell)}$ is zero. We check the $m_\ell = 2$ case. The other case is similar.

$$\begin{aligned}
U_\ell \phi_f^{(\ell)}(b) &= U_\ell\left(\epsilon(\ell)\phi_f(b)\right) \\
&= \sum_a \left(\left(\begin{smallmatrix} \ell & a \\ 0 & 1 \end{smallmatrix}\right) \circ \left(B^*_{\ell^2,1} - \ell^{-1} \cdot B^*_{\ell^2,\ell} \circ T_\ell + \ell^{-1} \cdot B^*_{\ell^2,\ell^2}\right)\right)\phi_f(b)
\end{aligned}$$

where $a = 0, \cdots, \ell - 1$. More computation shows that it coincides with:

$$\begin{aligned}
=&\sum_a \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}\left(b \cdot \left(\begin{smallmatrix} \ell & a \\ 0 & 1 \end{smallmatrix}\right)\right) - \frac{a_\ell}{\ell} \cdot \sum_a \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}\left(b \cdot \left(\begin{smallmatrix} \ell & \ell a \\ 0 & \ell \end{smallmatrix}\right)\right) + \frac{1}{\ell} \cdot \sum_a \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}\left(b \cdot \left(\begin{smallmatrix} \ell & \ell^2 a \\ 0 & \ell^2 \end{smallmatrix}\right)\right) \\
=&\sum_a \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}\left(b \cdot \left(\begin{smallmatrix} \ell & a \\ 0 & 1 \end{smallmatrix}\right)\right) - \frac{a_\ell}{\ell} \cdot \sum_a \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}\left(b \cdot \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right)\right) \\
&+ \frac{1}{\ell} \cdot \sum_a \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}\left(b \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a\ell \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right)\right)
\end{aligned}$$

Using $\phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}(b) = \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}\left(b \cdot \left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right)\right)$ and $\phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}(b) = \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}\left(b \cdot \left(\begin{smallmatrix} 1 & a\ell \\ 0 & 1 \end{smallmatrix}\right)\right)$, we continue the computation:

$$\begin{aligned}
=&\sum_a \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}\left(b \cdot \left(\begin{smallmatrix} \ell & a \\ 0 & 1 \end{smallmatrix}\right)\right) - a_\ell \cdot \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}(b) + \phi_f \circ \pi_{\ell,1} \circ \pi_{\ell,3}\left(b \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right)\right) \\
=&0.
\end{aligned}$$

Applying mod ϖ multiplicity one (Lemma 3.6), we obtain $\phi_{f^{(\ell)}} = u \cdot \phi_f^{(\ell)}$ for some $u \in \mathcal{O}$. Note that condition CR is used here. Thus, $\phi_f^{(\ell)}$ is normalized if and only if $u \in \mathcal{O}^\times$. Therefore, mod ϖ nonvanishing of $\phi_f^{(\ell)}$ implies the conclusion. □

In other words, we compare

$$\begin{array}{ccccc} f & \xrightarrow{\quad\ell\text{-deprivation}\quad} & f^{(\ell)} & \xrightarrow{\text{integral Jacquet-Langlands}} & \phi_{f^{(\ell)}} \\ & & & & \Big\| \text{up to a } p\text{-adic unit?} \\ f & \xrightarrow{\text{integral Jacquet-Langlands}} & \phi_f & \xrightarrow{\quad\ell\text{-deprivation}\quad} & \phi_f^{(\ell)} \end{array}$$

and we ask whether the latter one is also normalized.

If one iteratively applies Theorem 5.1, we obtain the following corollary.

Corollary 5.7. *Assume that $(\bar{\rho}, N^-)$ satisfies condition CR. Let $f = \sum_{n\geq 1} a_n q^n \in S_2(\bar{\rho}, N^-)$. Let Σ be a finite set of rational primes which split in K and are prime to p. Let f^{Σ} is the eigenform (of an appropriate level) whose q-expansion is*

$$\sum_{(n,\ell)=1, \ell\in\Sigma} a_n q^n.$$

Then we have

$$L_p(K_\infty, f^{\Sigma}) = u \cdot \left(\prod_{\ell\in\Sigma} \mathcal{E}_\ell(K_\infty, f) \right) \cdot L_p(K_\infty, f)$$

where $u \in \mathcal{O}^\times$. Moreover, we have

- $\mu(L_p(K_\infty, f^{\Sigma})) = \mu(L_p(K_\infty, f)) = 0$, *and*
- $\lambda(L_p(K_\infty, f^{\Sigma})) = \lambda(L_p(K_\infty, f)) + \sum_{\ell\in\Sigma} \lambda(\mathcal{E}_\ell(K_\infty, f))$.

Remark 5.8. (on the non-rational coefficient case in [GV00]) With the strategy of this section, [Wil95, Lemma 2.5], and the language of modular symbols, one can extend the results in [GV00] to the non-rational coefficient case without appealing to the delicate Hecke algebra argument in [EPW06].

5.2. **Mod ϖ nonvanishing of automorphic forms under stripping out Euler factors.** As we have seen, it suffices to consider the following question for a proof of Theorem 5.1.

Question 5.9. $\phi_f^{(\ell)} \not\equiv 0 \pmod{\varpi}$?

The following two theorems give an affirmative answer. The first theorem deals with the case $\ell \nmid N^+$ and the second one covers the case $\ell \| N^+$.

Theorem 5.10. *Let ℓ be a prime which splits in K and is prime to N^+. Let ϕ_f be an automorphic eigenform of weight 2, level N^+ and discriminant N^- (attached to a classical eigenform f). Let α_ℓ, β_ℓ be the roots of $x^2 - a_\ell(f)x + \ell$. Suppose that $\bar{\rho}_f$ is irreducible. Then $\phi_f^{(\ell)}$ are nonzero $\pmod{\varpi}$.*

Theorem 5.11. *Let ℓ be a prime which splits in K and $\ell \| N^+$. Let g be a classical eigenform of weight 2 and level N which is new at ℓN^-, and ϕ_g be the corresponding automorphic eigenform. Let a_ℓ be the U_ℓ-eigenvalue of g. Let $\phi_g^{(\ell)} := \phi_g \circ \pi_{\ell,3} - \frac{a_\ell}{\ell} \cdot \phi_g \circ \pi_{\ell,4}$ be the automorphic eigenform of level ℓN^+ whose U_ℓ-eigenvalue 0. Then $\phi_g^{(\ell)}$ is nonzero $\pmod{\varpi}$.*

Remark 5.12. (on the relation with μ-invariants) Question 5.9 can be interpreted in terms of μ-invariants. In this context, the answer means that their μ-invariants are *stable* under stripping out Euler factor at a prime which splits in K.

We give proofs of Theorem 5.10 and 5.11 in §5.4. The proofs depend heavily on Ihara's lemma.

5.3. **Ihara's lemma.** Let ℓ be a prime of $\mathbb{Q}$ such that $R_\ell^\times \simeq \mathrm{GL}_2(\mathbb{Z}_\ell)$. Let $R_0(\ell)^\times \subseteq R^\times$ be its Iwahori subgroup as in §2.2. The degeneracy map for the space of automorphic forms $M_2^{N^-}(K_0(N^+), \mathcal{O})$ is defined by

$$\begin{aligned} i_\ell : M_2^{N^-}(K_0(N^+), \mathcal{O}) \times M_2^{N^-}(K_0(N^+), \mathcal{O}) &\to M_2^{N^-}(K_0(\ell N^+), \mathcal{O}) \\ (\phi, \psi) &\mapsto \pi_{\ell,1}^*(\phi) + \pi_{\ell,2}^*(\psi) \end{aligned}$$

where $\pi_{\ell,1}^*(\phi) = \phi \circ \pi_{\ell,1}$ and $\pi_{\ell,2}^*(\psi) = \psi \circ \pi_{\ell,2}$.

Theorem 5.13. (Residual Ihara's lemma for automorphic forms: [DT94, Lemma 2]) *Let* $\mathfrak{m} \subseteq \mathbb{T}^{N^-}(K_0(N^+))_{\mathcal{O}}$ *be a non-Eisenstein maximal ideal. Then the ideal* $\mathfrak{m}$ *is not in the support of the kernel of the reduction of* i_ℓ*:*

$$i_{\ell,\mathbb{F}} : M_2^{N^-}(K_0(N^+),\mathbb{F}) \times M_2^{N^-}(K_0(N^+),\mathbb{F}) \to M_2^{N^-}(K_0(\ell N^+),\mathbb{F}).$$

We develop an analogue of [Wil95, Lemma 2.5] for the quaternionic setting. Let

- $\widehat{R^0(\ell)}^\times \subseteq \widehat{R}^\times$ where $R^0(\ell)_\ell^\times \subseteq \mathrm{GL}_2(\mathbb{Z}_\ell)$ is the lower triangular matrices mod ℓ and the prime-to-ℓ part coincides with that of $\widehat{R}^\times$ and
- $\widehat{R(\ell)}^\times \subseteq \widehat{R}^\times$ where $R(\ell)_\ell^\times \subseteq \mathrm{GL}_2(\mathbb{Z}_\ell)$ is the diagonal matrices mod ℓ and the prime-to-ℓ part coincides with that of $\widehat{R}^\times$.

Let

$$B_1 = \widehat{R_0(\ell)}^\times \Big/ \widehat{R(\ell)}^\times$$
$$B_2 = \widehat{R^0(\ell)}^\times \Big/ \widehat{R(\ell)}^\times$$
$$\Delta = \widehat{R}^\times \Big/ \widehat{R(\ell)}^\times.$$

Consider the natural isomorphisms

$$\lambda_1 : \mathrm{H}^0\left(\widehat{R_0(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right) \xrightarrow{\cong} \mathrm{H}^0\left(\widehat{R(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)^{B_1}$$
$$\lambda^1 : \mathrm{H}^0\left(\widehat{R^0(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right) \xrightarrow{\cong} \mathrm{H}^0\left(\widehat{R(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)^{B_2}$$
$$\mathrm{H}^0\left(\widehat{R}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right) \xrightarrow{\cong} \mathrm{H}^0\left(\widehat{R(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)^{\Delta}.$$

Then $\mathrm{im}(\lambda_1) \cap \mathrm{im}(\lambda^1) \subseteq \mathrm{H}^0\left(\widehat{R(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)^{\Delta}$ since Δ is generated by B_1 and B_2. Consider the sequence:

$$\begin{aligned}
0 \quad &\longrightarrow \quad \mathrm{H}^0\left(\widehat{R}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)\\
&\xrightarrow{\mathrm{res}_1\times -\mathrm{res}^1} \quad \mathrm{H}^0\left(\widehat{R_0(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right) \times \mathrm{H}^0\left(\widehat{R^0(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)\\
&\xrightarrow{\lambda_1+\lambda^1} \quad \mathrm{H}^0\left(\widehat{R(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)
\end{aligned}$$

where the map $\mathrm{res}_1 \times -\mathrm{res}^1$ is given by $\phi \mapsto (\mathrm{res}_1(\phi), -\mathrm{res}^1(\phi))$, and res_1, res^1 are the natural maps.

Proposition 5.14. *This sequence is exact.*

Proof. The injectivity of $\mathrm{res}_1 \times -\mathrm{res}^1$ is obvious. We compare the image of $\mathrm{res}_1 \times -\mathrm{res}^1$ and the kernel of $\lambda_1 + \lambda^1$. By definition of $\mathrm{res}_1 \times -\mathrm{res}^1$, we have (the image of $\mathrm{res}_1 \times -\mathrm{res}^1$) $\subseteq$ (the kernel of $\lambda_1 + \lambda^1$).

Let $\phi \in \mathrm{H}^0\left(\widehat{R_0(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)$ and $\psi \in \mathrm{H}^0\left(\widehat{R^0(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)$. Suppose that $\lambda_1(\phi) + \lambda^1(\psi) = 0$. Since $\lambda_1(\phi) = -\lambda^1(\psi)$, we know

$$\lambda_1(\phi) \in \mathrm{H}^0\left(\widehat{R(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)^{\Delta}.$$

Thus, $\lambda_1(\phi)$ comes from a unique element ϕ' of $\mathrm{H}^0\left(\widehat{R}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right)$. Then $\phi - \mathrm{res}_1(\phi')$ is in the kernel of λ_1 but $\ker(\lambda_1) = 0$, i.e. $\phi = \mathrm{res}_1(\phi')$. Thus, $\psi = -\mathrm{res}^1(\phi')$. $\square$

Proposition 5.15. *We have the following exact sequence*

$$\begin{aligned}
0 \quad &\longrightarrow \quad S_2^{N^-}(K_0(N^+),\mathbb{F})\\
&\longrightarrow \quad S_2^{N^-}(K_0(\ell N^+),\mathbb{F}) \times S_2^{N^-}(K_0(\ell N^+),\mathbb{F})\\
&\longrightarrow \quad S_2^{N^-}(K_0(\ell^2 N^+),\mathbb{F})
\end{aligned}$$

where the second arrow is $\phi \mapsto (\pi_{\ell,1}^*(\phi), -\pi_{\ell,2}^*(\phi)) = (\phi\circ\pi_{\ell,1}, -\phi\circ\pi_{\ell,2})$ *and the third arrow is* $(\psi_1,\psi_2) \mapsto \pi_{\ell,4}^*(\psi_1) + \pi_{\ell,3}^*(\psi_2)) = \psi_1 \circ \pi_{\ell,4} + \psi_2 \circ \pi_{\ell,3}$.

Proof. The idea comes from the conjugation appeared in in the proof of [Wil95, Lemma 2.5]. Consider isomorphisms

$$\begin{aligned} \mathrm{H}^0\left(\widehat{R^0(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right) &\to M_2^{N^-}(K_0(\ell N^+),\mathbb{F}) \\ \phi &\mapsto \phi^* \\ \mathrm{H}^0\left(\widehat{R(\ell)}^\times, \mathrm{Hom}(\mathbb{Z}[B^\times\backslash\widehat{B}^\times],\mathbb{F})\right) &\to M_2^{N^-}(K_0(\ell^2 N^+),\mathbb{F}) \\ \psi &\mapsto \psi^* \end{aligned}$$

where $\phi^*(b) := \phi(\mathrm{diag}(1,\ell^{-1})\cdot b\cdot \mathrm{diag}(1,\ell))$ and $\psi^*(b) := \psi(\mathrm{diag}(1,\ell^{-1})\cdot b\cdot\mathrm{diag}(1,\ell))$. Using these isomorphisms, we "twist" the exact sequence in Proposition 5.14 to obtain the exact sequence of the space of automorphic forms including the constant function, which is similar to the sequence in the statement. Also one can directly check that the morphisms in the twisted sequence become the ones in the statement. Excluding the constant function, we get the conclusion. □

5.4. **Proofs of Theorems 5.10 and 5.11.** In this section, we give an answer to Question 5.9 by proving Theorems 5.10 and 5.11.

Proof of Theorem 5.10. Nonvanishing of $\phi_f^{(\alpha_\ell)}$ and $\phi_f^{(\beta_\ell)} \pmod{\varpi}$ is a direct consequence of Theorem 5.13.

Suppose that $\phi_f^{(\ell)} \equiv 0 \pmod{\varpi}$. Since $\left(-\frac{1}{\beta_\ell}\cdot\phi_f^{(\alpha_\ell)}, \phi_f^{(\alpha_\ell)}\right)$ maps to $\phi_f^{(\ell)}$, Proposition 5.15 shows that $\left(-\frac{1}{\beta_\ell}\cdot\phi_f^{(\alpha_\ell)}, \phi_f^{(\alpha_\ell)}\right) = (\Phi\circ\pi_{\ell,1}, -\Phi\circ\pi_{\ell,2})$ for some $\Phi\in S_2^{N^-}(K_0(N^+),\mathbb{F})$. Since $\Phi\circ\pi_{\ell,1}$ is a constant multiple of $\phi_f^{(\alpha_\ell)}$, Φ itself is a mod ϖ eigenform for all Hecke operators at $q\neq\ell$. We have

$$\begin{aligned} \Phi\circ\pi_{\ell,1} &\equiv -\frac{1}{\beta_\ell}\cdot\phi_f^{(\alpha_\ell)} = -\frac{1}{\beta_\ell}\cdot\phi_f\circ\pi_{\ell,1} + \frac{1}{\ell}\cdot\phi_f\circ\pi_{\ell,2} \pmod{\varpi} \\ \Phi\circ\pi_{\ell,2} &\equiv -\phi_f^{(\alpha_\ell)} = -\phi_f\circ\pi_{\ell,1} + \frac{1}{\alpha_\ell}\cdot\phi_f\circ\pi_{\ell,2} \pmod{\varpi} \end{aligned}$$

as mod ϖ forms of level ℓN^+. Thus we have the congruence of level N^+ forms

$$\beta_\ell\cdot\Phi(B^\times\cdot b\cdot\widehat{R}^\times) \equiv \Phi(B^\times\cdot b\left(\begin{smallmatrix}1&0\\0&\ell\end{smallmatrix}\right)\cdot\widehat{R}^\times) \pmod{\varpi}.$$

We check Φ is an eigenform for T_ℓ. Applying T_ℓ to Φ, we have

$$\begin{aligned} (T_\ell\Phi)\left(B^\times\cdot b\cdot\widehat{R}^\times\right) &= \sum_{a=0}^{\ell-1}\Phi\left(B^\times\cdot b\left(\begin{smallmatrix}\ell&a\\0&1\end{smallmatrix}\right)\cdot\widehat{R}^\times\right) + \Phi(B^\times\cdot b\left(\begin{smallmatrix}1&0\\0&\ell\end{smallmatrix}\right)\cdot\widehat{R}^\times) \\ &\equiv \sum_{a=0}^{\ell-1}\Phi\circ\pi_1\left(B^\times\cdot b\left(\begin{smallmatrix}\ell&a\\0&1\end{smallmatrix}\right)\cdot\widehat{R_0(\ell)}^\times\right) + \beta_\ell\cdot\Phi(B^\times\cdot b\cdot\widehat{R}^\times) \pmod{\varpi} \\ &= U_\ell\left(\Phi\circ\pi_1\right)\left(B^\times\cdot b\cdot\widehat{R_0(\ell)}^\times\right) + \beta_\ell\cdot\Phi(B^\times\cdot b\cdot\widehat{R}^\times) \\ &\equiv U_\ell\left(-\frac{1}{\beta_\ell}\cdot\phi_f^{(\alpha_\ell)}\right)\left(B^\times\cdot b\cdot\widehat{R_0(\ell)}^\times\right) + \beta_\ell\cdot\Phi(B^\times\cdot b\cdot\widehat{R}^\times) \pmod{\varpi} \\ &= \alpha_\ell\cdot\left(-\frac{1}{\beta_\ell}\cdot\phi_f^{(\alpha_\ell)}\right)\left(B^\times\cdot b\cdot\widehat{R_0(\ell)}^\times\right) + \beta_\ell\cdot\Phi(B^\times\cdot b\cdot\widehat{R}^\times) \\ &\equiv \alpha_\ell\cdot(\Phi\circ\pi_1)\left(B^\times\cdot b\cdot\widehat{R_0(\ell)}^\times\right) + \beta_\ell\cdot\Phi(B^\times\cdot b\cdot\widehat{R}^\times) \pmod{\varpi} \\ &= \alpha_\ell\cdot\Phi\left(B^\times\cdot b\cdot\widehat{R}^\times\right) + \beta_\ell\cdot\Phi(B^\times\cdot b\cdot\widehat{R}^\times) \\ &= a_\ell\cdot\Phi\left(B^\times\cdot b\cdot\widehat{R}^\times\right). \end{aligned}$$

Thus, we obtain

$$(T_\ell \Phi)\left(B^\times \cdot b \cdot \widehat{R}^\times\right) = a_\ell \cdot \Phi\left(B^\times \cdot b \cdot \widehat{R}^\times\right).$$

The computation shows that Φ and ϕ_f have the same mod ϖ Hecke eigensystem. By mod ϖ multiplicity 1 (Lemma 3.6), we have $\Phi = u \cdot \phi_f$ for some $u \in \mathbb{F}^\times$. Then we have

$$u \cdot \phi_f \circ \pi_{\ell,1} = -\frac{1}{\beta_\ell} \cdot \phi_f \circ \pi_{\ell,1} + \frac{1}{\ell} \cdot \phi_f \circ \pi_{\ell,2}.$$

If $u = -\frac{1}{\beta_\ell}$, then $\frac{1}{\ell} \cdot \phi_f \circ \pi_{\ell,2}$ vanishes but it cannot happen due to the surjectivity of $\pi_{\ell,2}$. If $u \neq -\frac{1}{\beta_\ell}$, Ihara's lemma says that it cannot happen. Thus, $\phi_f^{(\ell)}$ does not vanish mod ϖ. $\square$

Proof of Theorem 5.11. Suppose that

$$\phi_g^{(\ell)} = \phi_g \circ \pi_{\ell,3} - \frac{a_\ell}{\ell} \cdot \phi_g \circ \pi_{\ell,4} \equiv 0 \pmod{\varpi}.$$

By Proposition 5.15, we have

$$\phi_g \equiv \Phi \circ \pi_{\ell,1} \pmod{\varpi}$$

for some $\Phi \in S_2^{N^-}(K_0(N^+), \mathbb{F})$. By the argument in the proof of Theorem 5.10, the mod ϖ form Φ admits a mod ϖ Hecke eigensystem. Thus, the corresponding residual Galois representation is unramified at ℓ. Note that Φ cannot be a form of level N^+/ℓ if $\overline{\rho}_\Phi$ is ramified at ℓ.

Now we may assume the residual representation is unramified at ℓ. Then by level lowering (*cf.* [Rib90b]), there exists an N^--new eigenform f of level $N/\ell = N^+N^-/\ell$ such that $\overline{\rho}_f \simeq \overline{\rho}_g$. Thus, the corresponding automorphic form ϕ_f is defined on $B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$, and it satisfies the congruence:

$$\phi_g \equiv u \cdot \phi_f^{(a_\ell)} \pmod{\varpi}$$

where $u \in \mathbb{F}^\times$ and a_ℓ is the U_ℓ-eigenvalue of g. This congruence implies that the proof reduces the proof of Theorem 5.10, so we get the conclusion. $\square$

6. The main theorem and its application to the anticyclotomic main conjecture

6.1. **The set of congruent newforms.** Let $(\overline{\rho}, N^-)$ be a pair satisfying condition CR (Definition 1.6) and $S_2(\overline{\rho}, N^-)$ be the set of p-ordinary newforms of weight 2 such that

- if $f \in S_2(\overline{\rho}, N^-)$, then $\overline{\rho} \simeq \overline{\rho}_f$.
- if $f \in S_2(\overline{\rho}, N^-)$, then $N^- = N^-_f$.
- if $f \in S_2(\overline{\rho}, N^-)$, then $a_p(f) \not\equiv \pm 1 \pmod{p}$.

The existence of infinitely many newforms in $S_2(\overline{\rho}, N^-)$ is ensured by a Chebotarev density type argument and [DT94].

6.2. **The main theorem.** The main theorem describes the behavior of analytic λ-invariants under congruences.

Theorem 6.1. (**Main Theorem**) *Suppose that $(\overline{\rho}, N^-)$ satisfies condition CR. Then we have the following:*

1. *$\lambda(L_p(K_\infty, f))$ is the same for all $f \in S_2(\overline{\rho}, N^-)$ such that $N(\rho_f)^+ = N(\overline{\rho})^+$; we call it $\lambda^{\mathrm{an}}(\overline{\rho})$.*
2. *For $f, g \in S_2(\overline{\rho}, N^-)$, we have*

$$\lambda(L_p(K_\infty, f)) + \sum_\ell \lambda(\mathcal{E}_\ell(K_\infty, f)) = \lambda(L_p(K_\infty, g)) + \sum_\ell \lambda(\mathcal{E}_\ell(K_\infty, g))$$

where the first and second sums run over the primes ℓ dividing $\mathrm{lcm}(N(\rho_f)^+, N(\rho_g)^+)$.

Before proving the main theorem, we prove the following theorem, which plays the key role in the proof.

Theorem 6.2. (Congruences between "L-values") *Suppose that $(\overline{\rho}, N^-)$ satisfies condition CR. Let f and g be N^--new eigenforms with the same residual representation $\overline{\rho}$. Further, assume that f and g are fully congruent. Then the following statements hold:*

(1) *The congruence between their anticyclotomic p-adic L-functions holds, i.e.*

$$L_p(K_\infty, f) \equiv u \cdot L_p(K_\infty, g) \pmod{\mathfrak{p}}$$

where $u \in \mathcal{O}^\times$.

(2) $\lambda(L_p(K_\infty, f)) = \lambda(L_p(K_\infty, g))$.

Proof. (1) Theorem 3.2 directly implies Theorem 6.2.(1) with definition of theta elements in §2.3.

(2) Because of vanishing of μ-invariants (Theorem 1.13), two mod p L-functions $L_p(K_\infty, f) \pmod{\mathfrak{p}}$ and $L_p(K_\infty, g) \pmod{\mathfrak{p}}$ encode their λ-invariants in the degree of (the reduction of) their distinguished polynomials. Hence, the conclusion follows.

□

Remark 6.3. (on the meaning of the congruence)

- The anticyclotomic p-adic L-function is well-defined only up to a p-adic unit due to the choice of normalization.
- With the interpolation formula, Theorem 6.2.(1) can be regarded as an anticyclotomic analogue of the weight 2 case of a theorem of Vatsal on congruences between modular L-values with cyclotomic twists in [Vat99].

Proof of Main Theorem (Theorem 6.1). Let f and g be arbitrary two forms in $S_2(\overline{\rho}, N^-)$. Let Σ be the set of rational primes dividing $\mathrm{lcm}(N(\rho_f)^+, N(\rho_g)^+)$. Consider fully Σ-deprived eigenforms f^Σ and g^Σ as in Corollary 5.7. Since their mod ϖ prime-to-Σ Hecke eigensystems coincide and their U_ℓ-eigenvalues are zero for all $\ell \in \Sigma$, f^Σ and g^Σ are *fully* congruent. Theorem 6.2.(1) implies that

$$L_p(K_\infty, f^\Sigma) \equiv u \cdot L_p(K_\infty, g^\Sigma) \pmod{\varpi}$$

where $u \in \mathcal{O}^\times$. Then, by Theorem 6.2.(2), we have the equality of λ-invariants

$$\lambda(L_p(K_\infty, f^\Sigma)) = \lambda(L_p(K_\infty, g^\Sigma)).$$

By Corollary 5.7, we have

$$\lambda(L_p(K_\infty, f)) + \sum_{\ell\in\Sigma} \lambda(\mathcal{E}_\ell(K_\infty, f)) = \lambda(L_p(K_\infty, g)) + \sum_{\ell\in\Sigma} \lambda(\mathcal{E}_\ell(K_\infty, g)).$$

For the first statement, assume that $f, g \in S_2(\overline{\rho}, N^-)$ with $N(\overline{\rho})^+ = N(\rho_f)^+ = N(\rho_g)^+$. Then we take Σ to be the set of rational primes dividing $N(\overline{\rho})^+ = N(\rho_f)^+ = N(\rho_g)^+$. Also, since $\deg(\mathcal{E}_\ell(K_\infty, f)) = \deg(\mathcal{E}_\ell(K_\infty, g))$ for all $\ell \in \Sigma$, we get the conclusion. □

We state the algebraic counterpart of the main theorem.

Notation 6.4. Write $\lambda(\mathrm{Sel}(K_\infty, A_f)) = \lambda(\mathrm{char}_\Lambda(\mathrm{Sel}(K_\infty, A_f))^\vee)$ for convenience.

Theorem 6.5. ([PW11, Theorem 7.1]) *Suppose that $(\overline{\rho}, N^-)$ satisfies condition CR and $\overline{\rho}$ has big image. Then we have*

(1) $\lambda(\mathrm{Sel}(K_\infty, A_f))$ *is the same for all $f \in S_2(\overline{\rho}, N^-)$ such that $N(\rho_f)^+ = N(\overline{\rho})^+$; we call it $\lambda^{\mathrm{alg}}(\overline{\rho})$.*

(2) *Let $f \in S_2(\overline{\rho}, N^-)$. Then we have*

$$\lambda(\mathrm{Sel}(K_\infty, A_f)) = \lambda^{\mathrm{alg}}(\overline{\rho}) + \sum_v \delta_v(f)$$

where the sum runs over the primes v of K_∞ dividing $N(\rho_f)^+$.

(3) *For two $f, g \in S_2(\overline{\rho}, N^-)$, we have*

$$\lambda(\mathrm{Sel}(K_\infty, A_f)) - \sum_v \delta_v(f) = \lambda(\mathrm{Sel}(K_\infty, A_g)) - \sum_v \delta_v(g)$$

where the first and second sums run over the primes v of K_∞ dividing $\mathrm{lcm}(N(\rho_f)^+, N(\rho_g)^+)$.

Remark 6.6. (on the algebraic counterpart)

(1) In [PW11], it is assumed that the level is squarefree; however, their work naturally extends to our setting without changing any argument.

(2) The algebraic counterpart requires the big image assumption since the Euler system divisibility is required to deduce vanishing of algebraic μ-invariants.

6.3. **Application to the anticyclotomic main conjecture.** We give an application of the main theorem (Theorem 6.1) to the main conjecture. Combining Theorem 1.9 and 6.5, we obtain the following Greenberg-Vatsal type result:

Corollary 6.7. (Application to the anticyclotomic main conjecture) *Suppose that* $(\overline{\rho}, N^-)$ *satisfies condition CR. Let* $f_1, f_2 \in S_2(\overline{\rho}, N^-)$.

(1) *Then we have*

$$\lambda(L_p(K_\infty, f_1)) - \lambda(\mathrm{Sel}(K_\infty, A_{f_1})) = \lambda(L_p(K_\infty, f_2)) - \lambda(\mathrm{Sel}(K_\infty, A_{f_2})).$$

(2) *Further, assume that* $\overline{\rho}$ *has big image. If the main conjecture holds for* one *form in* $S_2(\overline{\rho}, N^-)$, *then the main conjectures for* all *forms in* $S_2(\overline{\rho}, N^-)$ *hold.*

Proof. Let $f_1, f_2 \in S_2(\overline{\rho}, N^-)$. Then we have

$$\lambda(\mathrm{Sel}(K_\infty, A_{f_1})) - \sum_v \delta_v(f_1) = \lambda(\mathrm{Sel}(K_\infty, A_{f_2})) - \sum_v \delta_v(f_2)$$

by Theorem 6.5 and

$$\lambda(L_p(K_\infty, f_1)) + \sum_\ell \lambda(\mathcal{E}_\ell(K_\infty, f_1)) = \lambda(L_p(K_\infty, f_2)) + \sum_\ell \lambda(\mathcal{E}_\ell(K_\infty, f_2))$$

by the main theorem (Theorem 6.1). Thus, we have

$$\begin{aligned} &\lambda(L_p(K_\infty, f_1)) - \lambda(\mathrm{Sel}(K_\infty, A_{f_1})) + \sum_\ell \lambda(\mathcal{E}_\ell(K_\infty, f_1)) + \sum_v \delta_v(f_1) \\ =&\lambda(L_p(K_\infty, f_2)) - \lambda(\mathrm{Sel}(K_\infty, A_{f_2})) + \sum_\ell \lambda(\mathcal{E}_\ell(K_\infty, f_2)) + \sum_v \delta_v(f_2). \end{aligned}$$

Since two terms $\sum_\ell \lambda(\mathcal{E}_\ell(K_\infty, f_1)) + \sum_v \delta_v(f_1)$ and $\sum_\ell \lambda(\mathcal{E}_\ell(K_\infty, f_2)) + \sum_v \delta_v(f_2)$ depend only on the residual representation (Theorem 4.8), they coincide. Therefore, by Theorem 1.9, conclusion (1) follows. Combining with the Euler system divisibility (Theorem 1.9), we have conclusion (2). ☐

Remark 6.8. (Comparison with the work of Skinner-Urban [SU14]) In [SU14], they proved the three variable Iwasawa main conjecture for a large class of modular forms. See [SU14, Theorem 3.37] for the anticyclotomic specialization. In [SU14, §2.1.1], it is assumed that p splits in K. Also, *Hida's canonical periods* are used in the interpolation formula [SU14, §3.4.5.(3.1.3)], and vanishing of the anticyclotomic μ-invariants with Hida's canonical periods is crucially used in the proof of [SU14, Proposition 12.9].

In our work, *Gross' periods* are used for the normalization, and Gross' and Hida's ones are *different* in general. They coincide (up to a p-adic unit) if and only if the residual representation $\overline{\rho}$ is ramified at *all* prime divisors of N^-. For more detail, see [PW11, Theorem 2.3].

We end this section with a simple corollary.

Corollary 6.9. *Under the same assumptions as in Corollary 6.7.(2), if there exists a form* $f \in S_2(\overline{\rho}, N^-)$ *such that* $L(K_\infty, f) \in \Lambda^\times$ *(equivalently,* $\lambda^{\mathrm{an}}(\overline{\rho}) = 0$*), then the main conjecture holds for* all *forms in* $S_2(\overline{\rho}, N^-)$.

7. An explicit example

All the computation is done with [S+13].

Let $p = 5$. Consider an elliptic curve E over $\mathbb{Q}$ defined by the following (minimal) equation:

$$E : y^2 + y = x^3 + x^2 + 4033x - 337456.$$

Then we have the following information:

- $j(E) = \frac{2^{15}}{3^5}$.
- The root number of $E = 1$.
- $E[5]$ is surjective.
- $a_5(E) = 2$, so ordinary at 5 but not anomalous.
- $\mathrm{Tam}_3(E) = 5$. (See [PW11, Definition 3.3]) for the definition.)
- $N = N_E = 109771203 = 3 \cdot 23^2 \cdot 263^2$.
- $\Delta_E = -53784486562707 = -1 \cdot 3^5 \cdot 23^3 \cdot 263^3$.

Let $K = \mathbb{Q}(\sqrt{-19})$ so that $N^- = 3$ and $N^+ = 23^2 \cdot 263^2$. Note that 5 splits in K. Then $E[5]$ is *unramified* at 3 by the Tate curve argument for $E/\mathbb{Q}_3$. Thus, Skinner-Urban's result does *not* apply in this situation. Condition CR is satisfied since $3 \not\equiv 1 \pmod 5$.

Let χ be the quadratic character associated to K. Then the quadratic twist of E by χ is given by the following (minimal, again) equation:

$$E^\chi : y^2 + y = x^3 - x^2 + 1455793x - 2323343999.$$

Then we have the following L-values:

$$\frac{L(E,1)}{\Omega^+_E} = 20 = 2^2 \cdot 5$$
$$\frac{L(E^\chi,1)}{\Omega^+_{E^\chi}} = 4 = 2^2$$

where Ω^+_E and $\Omega^+_{E^\chi}$ are their real Néron periods, respectively.

We show the calculation and explain each step. The relation $\sim$ means that they are equal up to a 5-adic unit.

$$\begin{aligned}
2^4 \cdot 5 = \frac{L(E,1)}{\Omega^+_E} \cdot \frac{L(E^\chi,1)}{\Omega^+_{E^\chi}} &\sim \frac{L(E,1)}{\Omega^+_E} \cdot \frac{L(E^\chi,1)}{\Omega^-_E} \sim \frac{L(E,1)}{-2\pi i \Omega^+_{f_E}} \cdot \frac{L(E^\chi,1)}{-2\pi i \Omega^-_{f_E}} \\
&\sim \frac{L(E,1)\cdot L(E^\chi,1)}{\frac{\langle f,f\rangle_{\Gamma_1(N)}}{\eta_{\Gamma_1(N)}}} \sim \frac{L(E,1)\cdot L(E^\chi,1)}{\frac{\langle f,f\rangle_{\Gamma_0(N)}}{\eta_{\Gamma_1(N)}}} \sim \frac{L(E,1)\cdot L(E^\chi,1)}{\frac{\langle f,f\rangle_{\Gamma_0(N)}}{\eta_N}\frac{\eta_N}{\eta_{\Gamma_1(N)}}} \\
&\sim \frac{L(E,1)\cdot L(E^\chi,1)}{\frac{\langle f,f\rangle_{\Gamma_0(N)}}{\eta_N}} \cdot \frac{\eta_{\Gamma_1(N)}}{\eta_N} \sim 5 \cdot \frac{L(E,1)\cdot L(E^\chi,1)}{\frac{\langle f,f\rangle_{\Gamma_0(N)}}{\eta_{N^+,N^-}}} \cdot \frac{\eta_{\Gamma_1(N)}}{\eta_N}.
\end{aligned}$$

The first equality is just SAGE calculation. The second line to compare the periods $\Omega^+_{E^\chi}$ and Ω^-_E follows from the formula given in [Pal12]. The third equivalence comes from [GV00, Proposition 3.1 and Remark 3.4]. The fourth equivalence is a property of Hida's canonical periods for $\Gamma_1(N)$. See Appendix A and also [Vat03, Remark 2.7] and [Wil95, Proposition 4.4 and 4.5]. Since the index $[\Gamma_0(N) : \Gamma_1(N)] = \phi(109771203) = 69732872$ is prime to 5, the fifth one trivially holds. In the sixth one, η_N is the congruence number for $\Gamma_0(N)$. Since $\Gamma_1(N) \subseteq \Gamma_0(N)$, we have $S_2(\Gamma_0(N)) \subseteq S_2(\Gamma_1(N))$, so $\eta_N \mid \eta_{\Gamma_1(N)}$. The fact $\eta_N \mid \eta_{\Gamma_1(N)}$ implies that $\frac{\eta_{\Gamma_1(N)}}{\eta_N}$ is integral. The sixth and seventh ones follow from this. The last equivalence follows from the μ-invariant formula of Pollack-Weston ([PW11, Theorem 6.8]). The calculation implies that the L-value

$$\frac{L(E,1)\cdot L(E^\chi,1)}{\frac{\langle f,f\rangle_{\Gamma_0(N)}}{\eta_{N^+,N^-}}}$$

is a 5-adic unit. Using the interpolation formula, we obtain the λ-invariant of $L_p(K_\infty, E) = L_p(K_\infty, f)$ is 0. Thus, the anticyclotomic main conjecture for this elliptic curve trivially holds. Then, by Corollary 6.7, the anticyclotomic main conjectures for *all* forms in $S_2(E[5], 3)$ hold although many forms in $S_2(E[5], 3)$ have nonzero λ-invariants.

Appendix A. Comparison of periods

The aim of this section is to compare various normalizations of the periods which appeared in the literature. We thank Nike Vatsal for his encouragement to write this appendix.

A.1. **Gross' periods.**

A.1.1. *Gross.* ([Gro87, Page 148]) Let $f(z)$ be a (normalized) newform of weight 2 and $\Gamma_0(N)$ where N is a prime. Let $\omega_f = 2\pi i f(z)dz = \sum_{m\geq 1} a_m q^m \frac{dq}{q}$ be the holomorphic differential associated to $f(z)$

on $X_0(N)$. Then the Gross' normalization of period is given as follows:

$$\begin{aligned}(f,f)_{\mathrm{Gross}} &= (f,f)_{\Gamma_0(N)} \\ &:= \int\int_{X_0(N)} \omega_f \wedge (i\cdot\overline{\omega_f}) \\ &= 8\pi^2 \int\int_{\Gamma_0(N)\backslash\mathfrak{h}} f(z)\overline{f(z)}dxdy\end{aligned}$$

Note that no congruence number appears in Gross' original formula since N is prime.

A.1.2. *Vatsal and Pollack-Weston.* Vatsal's normalization follows Gross' one. Although the explicit normalization of the Petersson inner product is not given in [Vat03], the special value formulas of Gross and Vatsal coincide when the character is unramified and $\mathcal{O}_K^\times = \{\pm 1\}$. Thus, we deduce $(f,f)_{\mathrm{Vatsal}} = (f,f)_{\mathrm{Gross}}$. See [Vat03, (2.3) and (2.4)] and [Gro87, Proposition 11.2]. **Gross' period** is defined by

$$\Omega^{\mathrm{Gross}}_{f,N^-} := \frac{(f,f)_{\mathrm{Gross}}}{\xi_f(N^+,N^-)}$$

where $\xi_f(N^+,N^-)$ is defined in [PW11, §2]. In [PW11, §6.6], it is proved that the quantity $\xi_f(N^+,N^-)$ and the corresponding congruence number coincide up to a p-adic unit under condition CR.

Remark A.1. In [CH16], an *half* of $\Omega^{\mathrm{Gross}}_{f,N^-}$ is used in the interpolation formula. However, it does not cause any problem because $\xi_f(N^+,N^-)$ is defined only up to a p-adic unit and p is odd.

A.2. **Hida's canonical periods and congruence numbers.**

A.2.1. $\Gamma_0(N)$. In [Vat03, §2.4], **Hida's canonical periods for** $\Gamma_0(N)$ is defined by

$$\Omega^{\mathrm{can}}_f := \frac{(f,f)_{\mathrm{Gross}}}{\eta_f(\Gamma_0(N))}$$

where $\eta_f(\Gamma_0(N))$ is the congruence number of f in $S_2(\Gamma_0(N))$.

A.2.2. $\Gamma_1(N)$. In [Vat03, Remark 2.7], **Hida's canonical periods for** $\Gamma_1(N)$ is defined by

$$\Omega^{\mathrm{Hida}}_f := \frac{(f,f)_{\Gamma_1(N)}}{\eta_f(\Gamma_1(N))}$$

where $\eta_f(\Gamma_1(N))$ is the congruence number of f in $S_2(\Gamma_1(N))$.

The Petersson inner product for $\Gamma_1(N)$ is normalized as follows:

$$\begin{aligned}(f,f)_{\Gamma_1(N)} &= 8\pi^2\cdot[\Gamma_0(N):\Gamma_1(N)]\cdot\int\int_{\Gamma_0(N)\backslash\mathfrak{h}} f(z)\overline{f(z)}dxdy \\ &= \phi(N)\cdot(f,f)_{\mathrm{Gross}}\end{aligned}$$

where ϕ is the Euler totient function. We also have

$$\Omega^{\mathrm{Hida}}_f = \frac{(f,f)_{\Gamma_1(N)}}{\eta_f(\Gamma_1(N))} = \Omega^+_f\cdot\Omega^-_f$$

where $\Omega^{\pm}_f$ are the canonical periods given in [Vat99] and [Vat13]. For a proof of the last decomposition, see [Wil95, Proposition 4.4 and 4.5].

A.3. **Gross' periods revisited.** We give another description of Gross' periods as in [CH16]. We follow the notation in [CH16] and focus on the case of weight 2. Their Gross' period in the interpolation formula [CH16, Theorem A] is defined by

$$\Omega_{f,N^-} := \frac{4\pi^2\|f\|_{\Gamma_0(N)}}{\xi_f(N^+,N^-)}.$$

This period is calculated in terms of automorphic representations as follows ([CH16, (4.3)]):

$$\Omega_{\pi,N^-} := \frac{4\pi^2\|\varphi_\pi\|_{\Gamma_0(N)}}{\langle f_{\pi'},f_{\pi'}\rangle_R}.$$

It is known that $\xi_f(N^+,N^-) = \langle f_{\pi'}, f_{\pi'}\rangle_R$ (up to a p-adic unit) due to [PW11]. This quantity comes from the normalization of automorphic forms on a definite quaternion algebra. The term $\|\varphi_\pi\|_{\Gamma_0(N)}$ is given as follows ([CH16, §3.3]):

$$\|\varphi_\pi\|_{\Gamma_0(N)} := \mathrm{vol}(U_0(N), d^t g)^{-1} \cdot \int_{\mathbb{A}^\times \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})} \mid \varphi_\pi(g) \mid^2 \, d^t g$$

where $d^t g$ is the Tamagawa measure on GL_2, and $U_0(N) = \mathrm{O}(2,\mathbb{R}) \times \prod_{q<\infty} U_0(N)_q$. The normalization of the Tamagawa measure is given by

$$\mathrm{vol}(\mathbb{A}^\times \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}), d^t g) = 2.$$

Note that

$$\begin{aligned}\mathrm{vol}(U_0(N), d^t g)^{-1} &= \zeta_{\mathbb{Q}}(2) N \prod_{q\mid N}(1+q^{-1}) = \frac{1}{\pi}\cdot\frac{\pi^2}{6}\cdot[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]\\ &= \frac{\pi}{6}\cdot[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)] = \frac{1}{2}\cdot \mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}, \frac{dxdy}{y^2})\cdot[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]\end{aligned}$$

where $\zeta_{\mathbb{Q}}(s)$ is the complete Riemann zeta function so that

$$\zeta_{\mathbb{Q}}(2) = \zeta_{\mathbb{R}}(2)\cdot\zeta(2) = \frac{1}{\pi}\cdot\frac{\pi^2}{6}.$$

Then we have

$$\begin{aligned}\|f_\pi\|_{\Gamma_0(N)} &= \|\varphi_\pi\|_{\Gamma_0(N)}\\ &= \frac{1}{2}\cdot \mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}, \frac{dxdy}{y^2})\cdot[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]\cdot\int_{\mathbb{A}^\times \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})} \mid \varphi_\pi(g) \mid^2 \, d^t g.\end{aligned}$$

Here, the normalization in LHS is

$$\|f_\pi\|_{\Gamma_0(N)} = \int\int_{\Gamma_0(N)\backslash\mathfrak{h}} f_\pi(z)\overline{f_\pi(z)}dxdy$$

so that

$$\|f_\pi\|_{\Gamma_0(N)} = \frac{1}{8\pi^2}\cdot(f_\pi, f_\pi)_{\mathrm{Gross}}.$$

This normalization explains Remark A.1, i.e.

$$\Omega^{\mathrm{Gross}}_{f,N^-} = \frac{(f,f)_{\mathrm{Gross}}}{\xi_f(N^+,N^-)} = 2\cdot\Omega_{f,N^-} = 2\times\frac{4\pi^2\|f\|_{\Gamma_0(N)}}{\xi_f(N^+,N^-)}.$$

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