A VERY STRANGE GROUP COHOMOLOGY

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Abstract. We ask a question on the structure of certain group cohomology modules of *countably infinitely generated* profinite groups with coefficient modules of *countably infinite* rank.

1. The statement of the question

Let p be a (sufficiently large if needed) prime. Let G be a countably infinite product of finite cyclic groups of p-power order, so it is an infinitely generated profinite group. The power of p is not necessarily bounded in G. Let M be a discrete G-module of countably infinite $\mathbb{Z}$-rank with finite prime-to-p torsions such that M^G is finitely generated. We have an isomorphism as abelian groups

$$M \simeq \mathbb{Z}^{\oplus\infty} \oplus M_{\mathrm{tors}}$$

where M_{tors} is a finite abelian group of prime-to-p order. Since $M[p] = 0$, we have an exact sequence

$$0 \longrightarrow M \xrightarrow{p^k} M \longrightarrow M/p^kM \longrightarrow 0$$

for every $k \geq 1$. By taking the G-invariant, we have an exact sequence

$$0 \longrightarrow M^G/p^kM^G \longrightarrow \left(M/p^kM\right)^G \longrightarrow \mathrm{H}^1(G, M)[p^k] \longrightarrow 0.$$

By taking the direct limit, we have an exact sequence

$$0 \longrightarrow M^G \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \varinjlim_k \left(M/p^kM\right)^G \longrightarrow \mathrm{H}^1(G, M)[p^\infty] \longrightarrow 0.$$

Question 1.1. How does

$$\mathrm{H}^1(G, M)[p^\infty]$$

look like and how much can it be big? Which kinds of further conditions are needed to make it smaller? For example, can we determine

$$\mathrm{cork}_{\mathbb{Z}_p}\mathrm{H}^1(G, M)[p^\infty]$$

or make it zero under certain circumstances?

Feel free to contact me if you observe anything regarding the above question.

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